Robust Reconstruction of Nonlinear Models Parameters From Measurement Data

M.Sc Proposal

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Abstract

The first part of this work presents several results about non linear reconstruction of Fourier transform for "simple" functions. The second and the main part of this work deals with the Colliding Target Detection problem (CTD). This problem concerns detection of two or more targets which are close to each other. The usual resolution in this case drops, so we shall analyze this problem utilizing the simplicity of the transmitted and received signals. From the CTD problem as well from the non linear Fourier transform reconstruction arises the same set of non linear equations. We shall show some directions for solving it.

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Contents

1 Summary.

1.1 The approach.

Consider the following general situation: we have to restore from a set of noisy measurements a certain “signal” belonging to a given class (CT images, radio signals of various origins, etc.). Assume that we are given a *non-linear approximation scheme S for signals from our class, which allows for an accurate representation of each specific signal, and whose parameters reflect the features we are interested in.* We shall denote by $S(\lambda)$ the signal model provided by S, where λ is the set of the parameters.

In many cases such a scheme S is suggested by the mere physical nature of the signal considered, like in the case of the reflected radar signal, which is the superposition of the original pulse reflected from different targets. In this case the most important non-linear parameters of S are the targets positions. In other cases, the construction of the scheme S is not easy, as in the case of medical and other images.

The main problem considered in this work is the following: *given a class of signals and a non-linear approximation scheme S for signals from our class, how to reconstruct with a high accuracy the parameters λ of S from a set of noisy measurements of the actual signal? In particular in this work we shall focus on the colliding targets detection problem and bring up some methods towards its solution.*

Our basic approach is the following: Using specific mathematical features of the problem we construct a system of non linear equations. This system's solution assumed to be the “best fitting” parameters λ (scalar or vector) of S s.t the actual measurements and the measurements of the model $S(\lambda)$ are ”close” to each other (”closeness” is to be discussed later).

We address in this thesis two different but related problems: The Fourier inversion on piecewise polynomials and Radar target detection problem.

1.2 The nonlinear Fourier inversion problem.

We shall sketch an approach for the investigation of the general problem of nonlinear parameters reconstruction with some specific problems, related to the reconstruction from the Fourier data of functions with the "edge-type" singularities. The approximating models $S(\lambda)$ are piecewise polynomials. In this initial stage we assume all the data to be accurate and do not address explicitly the problems of the noise separation. This approach is for future investigation and it will not present a significant part in the results and the conclusions of this work.

1. Reconstruction of a piecewise constant function from its Fourier coefficients.

In fact, we replace the Fourier coefficients by the moments, and obtain explicitly a system of nonlinear equations for the coordinates of the discontinuity points and for the function values between the jumps. We shall show that also *the solution of this system can be obtained explicitly.* (see §4.7 below)

2. Reconstruction of a piecewise polynomial function from its Fourier coefficients.

It looks plausible that in the case of a piecewise polynomial function it is still possible to solve the corresponding system of nonlinear equations explicitly. The corresponding nonlinear systems of equations are closely related to the ones we found in the target detection problem. We shall describe these systems below.

1.3 The Radar Target Detection problem.

We suggest to start the investigation of the nonlinear reconstruction in the case of the noisy data from a very specific problem, arising in the analysis of the radar signals. In some aspects this problem has a simpler formulation than the general reconstruction problem above. On the other hand, this problem is characterized by a very high noise level, it is still open in some important instances, and it is practically important.

1. The simplified Target Detection problem.

Here we assume that the reflected radar signal is the sum of the original pulse, reflected from different point targets. We assume in addition that the amplitude and the initial phase of the reflected pulses are preserved. This last assumption is not realistic, and we remove it later on.

Under the above assumptions the only parameters of S are the targets positions, i.e. the time shifts of the reflected pulses.

We propose to investigate this problem on the base of convolving the Taylor polynomial of the signal against a precalculated kernels which diagonalize the system of moments. This Taylor polynomial is developed w.r.t the targets positions. This method gives a system of nonlinear equations that approximate our parameters. .

2. The extended Target Detection problem.

As above, we assume that the reflected radar signal is the sum of the original pulse, reflected from different point targets. However, as it happens in reality, the amplitude and the initial phase of the reflected pulses may vary, and they are not known a priori.

We get more free parameters than in the simplified problem above. We propose to estimate these parameters on the base of a more careful analysis of the correlation function.

3. The problem of detection of colliding targets.

The problem of an accurate detection of two or more targets roughly at the same distance is practically very important. The resolution of the conventional methods drops in such situations. We propose to investigate this problem combining the analysis of the convolution function with the Taylor expansion of the signal with respect to a small parameter (or parameters).

4. An analytic method to solve the nonlinear system of moments equation

We shall discuss a method given by [7] to solve the very special form of equations that arise from the colliding targets detection problem, as well from the Fourier transform reconstruction of piecewise constant functions.

2 Fourier transform of piecewise polynomial functions.

In this section we study the behavior of the Fourier transform on the classes of piecewise polynomial functions of the prescribed *combinatorial* complexity. Let us introduce some notations. We consider the space $L^2(S^1)$ of all the square integrable functions on the unit circle S^1. The Fourier transform is a mapping (an isomorphism) $F : L^2(S^1) \to l^2$ of $L^2(S^1)$ into the space l^2 of all square summable double infinite sequences $(...a_{-n}, a_{-n+1}, \ldots, a_{n-1}, a_n, \ldots)$, defined by $a_n(f) = \frac{1}{2\pi}\int_{S^1} f(t)\exp(-int)\,dt$. We shall use also a partial Fourier transform F_k which associates to a function f its Fourier polynomial of degree k, i.e: $F_k(x) = \sum_{n=-k}^{k} a_n(f)\,e^{inx}$.

2.1 Complexity of piecewise polynomial functions.

We consider piecewise polynomial functions on S^1. Let such a function $g(x)$ be represented by the polynomials $P_q(x)$ on the intervals $\Delta_q,\ q = 1, ..., r$ of the partition Σ of S^1. We define the *combinatorial complexity* $\sigma_{PP}(g)$ of g as follows:

Definition 2.1 *The combinatorial complexity $\sigma_{PP}(g)$ is the sum $r + \sum_{q=1}^{r} d_q$ or $\sum_{q=1}^{r}(d_q + 1)$.*

The specific choice of this complexity expression is motivated by the following simple observation:

Proposition 2.1 *The number of sign changes of a piecewise polynomial function g on S^1 does not exceed $\sigma_{PP}(g)$.*

Proof: Sign changes of g may occur either at the end points of the intervals $\Delta_q,\ q = 1, ..., r$ of the partition Σ of S^1, or at the interior points of these intervals Δ_q at which the polynomial P_q vanishes. The total number of zeroes of all the polynomials P_q does not exceed $\sum_{q=1}^{r} d_q$. Adding the endpoints of Δ_q we get the required expression.

We need also the following simple lemma:

Lemma 2.1 *Let $g_1, \ldots, g_l$ be piecewise polynomial functions with $\sigma_{PP}(g_j) \le \mathrm{d}_j,\ j = 1, \ldots, l$. Then for $g = g_1 + \ldots + g_l$ the combinatorial complexity $\sigma_{PP}(g)$ satisfy*

$$\sigma_{PP}(g) \le \delta(\delta+1),$$

where $\delta = (d_1 + \ldots + d_l)$.

Proof: Consider a partition Σ of S^1 which is a common refinement of all the partitions $\Sigma_j,\ j = 1, \ldots, l$ of g_j. The number of the intervals in each Σ_j does not exceed d_j. Therefore, the number of the intervals Δ_q in Σ does not exceed $d_1 + \ldots + d_l = \delta$. g is a polynomial on each Δ_q, of the degree at most equal the maximum of the degrees of the polynomials forming g_j. In particular, this degree does not exceed $d_1 + \ldots + d_l = \delta$. By definition 2.1, we have then $\sigma_{PP}(g) \le \delta(\delta+1)$. This completes the proof of the lemma.

Remark 1. The bound of Lemma 2.1 is essentially sharp. Indeed, consider the case where g_1 is a polynomial of degree d_1, while g_2 is a piecewise constant function with d_2 partition intervals. We have $\sigma_{PP}(g_1) = d_1 + 1,\ \sigma_{PP}(g_1) = d_2$. The sum $g = g_1 + g_2$ is a piecewise polynomial function with d_2 partition intervals, and with the degree of the polynomial on each interval equal to d_1. Hence $\sigma_{PP}(g) = (d_1 + 1)\, d_2$. For large and roughly equal d_1, d_2 this is of the same order as the bound of Lemma 2.1.

Remark 2. Notice, however, that *the number of zeroes and of sign changes* of g in the example above is bounded by $d_2 + d_1$. Indeed, let us count the sign changes of g along the segments where g_1 is monotone. On each segment the sign changes are bounded by the number of partitioning intervals overlapping this segment. All together we counted the number of the partitioning intervals while counting twice each interval that has an extremum of p in it's interior. All together we can not have more than $d_1 - 1 + d_2$. Recalling that if the polynomial is of odd degree, sign change can occur also at the end and beginning point of S^1 gives us $d_1 + d_2$.

2.2 Injectivity of the Fourier transform.

Let $PP(\delta) \subset L^2(S^1)$ be the subset of all square-integrable semi-algebraic functions of the combinatorial complexity (as defined in §2.1 above) at most δ.

Notice that $PP(\delta)$ *is not a linear subspace* in $L^2(S^1)$. Indeed, $PP(\delta)$ contains, in particular, all the piecewise constant functions with the number of the partition intervals at most δ. However, for a generic choice of the partition points the sum $f+g$ of two such functions f and g is piecewise constant only on the refined partition, containing 2δ intervals, and hence its combinatorial complexity $\sigma_{PP}(f+g)$ is 2δ. In general, for $f_1,\ldots,f_l$ piecewise constant functions with the number of the partition intervals $\delta_1,\ldots,\delta_l$, respectively, we have $\sigma_{PP}(f_1+\ldots+f_l) \le \delta_1+\ldots+\delta_l$, and the equality holds for a generic choice of the partitions. In general, the behavior of a combinatorial complexity of piecewise polynomial functions is partially described by Lemma 2.1.

Now we can prove our first main result, showing that a piecewise polynomial function of a combinatorial complexity δ is uniquely defined by its first $r = 2\delta(2\delta+1)$. By first r Fourier coefficients we mean $\{a_k(f)\}_{k=-\frac{r}{2}}^{\frac{r}{2}}$. Fourier coefficients. We do not touch in this stage the question of *how such a function can be actually reconstructed from the Fourier data.*

We shall prove two theorems, one for moment transform and another for Fourier transform. Both of them use the same basic method while the first theorem is maybe less technical.

First we shall consider the moments $m_k(g) = \int_0^{2\pi} t^k g(t)\,dt$ $k = 0,1,\ldots$. So we consider the "moment transform"

$$M(g) = (m_0(g), m_1(g), \ldots, m_r(g), \ldots),$$

and the partial moment transforms

$$M_r(g) = (m_0(g), m_1(g), \ldots, m_r(g)).$$

We shall prove injectivity of the appropriate moment transform on $PP(\delta)$ and then return to the original setting of the Fourier transform.

Theorem 2.1 *M_r is injective on $PP(\delta)$.*

Proof: Assume, in contrary, that there are functions g_1 and g_2 in $PP(\delta)$, $g_1 \ne g_2$ with exactly the same moments up to r-th. Hence for the difference $g = g_2 - g_1 \ne 0$ we have the vanishing of the moments up to r-th: $m_j(g) =$

$0,\ j = 0, 1, \ldots, r$. By Lemma 2.1 we have for the combinatorial complexity of g the bound $\sigma_{PP}(g) \leq r$. By Proposition 2.1 $\sigma_{PP}(g)$ bounds from above the number of sign changes of g. Thus g changes sign at certain points $x_1, \ldots, x_q,\ q \leq r$ (If g flip sign by vanishing on some interval we take only one point -w.l.o.g the middle- in this interval as x_i).

Let us construct now a polynomial $Q(t)$ of degree q with exactly the same sign pattern as g that is: $Q(t) = \operatorname{sgn}(g(0))(-1)^q \prod_{i=1}^{q}(x - x_i)$. Write Q as $Q(x) = \sum_0^q \alpha_j x^j$. We have $g(x)Q(x) > 0$ everywhere, except the points $x_1, \ldots, x_q$ and the intervals where $g \equiv 0$ (which are not all the domain since $g \not\equiv 0$). Therefore $\int_{S^1} g(x)Q(x) > 0$. On the other hand, this integral can be expressed as a linear combination of the moments: $0 < \int_{S^1} g(x)Q(x) = \sum_1^q \alpha_j \int_{S^1} x^j g(x)\,dx = \sum_0^q \alpha_j m_j(g) = 0$, since all the moments of g up to r-th vanish by the assumption. This contradiction proves Theorem 2.1.

Lemma 2.2 *For even q and $\{x_i\}_{i=1}^{q} \subset \mathbb{C}$ the product*

$$\prod_{i=1}^{q} \sin\left(\frac{1}{2}(x - x_i)\right) \tag{1}$$

can be expressed as a trigonometric polynomial

$$\sum_{j=-\frac{q}{2}}^{\frac{q}{2}} \alpha_j e^{ijx}. \tag{2}$$

Proof: In the preceding calculation we will use the trigonometric identities:

$$2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$$

and

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta),$$

and the Euler identity.

$$\begin{array}{l}\prod_{i=1}^{q}\sin\left(\frac{1}{2}(x-x_i)\right)\\ =\prod_{j=1}^{\frac{q}{2}}\sin\left(\frac{1}{2}(x-x_{2j-1})\right)\sin\left(\frac{1}{2}(x-x_{2j})\right)\\ =\frac{1}{2}\prod_{j=1}^{\frac{q}{2}}\left\{\cos[\frac{1}{2}(x-x_{2j-1})-\frac{1}{2}(x-x_{2j})]-\cos[\frac{1}{2}(x-x_{2j-1})+\frac{1}{2}(x-x_{2j})]\right\}\\ =\frac{1}{2}\prod_{j=1}^{\frac{q}{2}}\cos\left(-\frac{1}{2}x_{2j-1}+\frac{1}{2}x_{2j}\right)-\cos\left(x-\frac{1}{2}x_{2j-1}-\frac{1}{2}x_{2j}\right)\\ =:\frac{1}{2}\prod_{j=1}^{\frac{q}{2}}a_j-\cos(x+b_j)=\frac{1}{2}\prod_{j=1}^{\frac{q}{2}}a_j-\cos x\cos b_j+\sin x\sin b_j\\ =:\prod_{j=1}^{\frac{q}{2}}\tilde{a}_j+\tilde{b}_je^{ix}+\tilde{\gamma}_je^{-ix}=\sum_{j=-\frac{q}{2}}^{\frac{q}{2}}\alpha_je^{ijx}\end{array} \tag{3}$$

Using the fact that $e^{inx}e^{imx}=e^{i(n+m)x}$, the last equality is easy to verify. By this we proved Lemma 2.2.

Theorem 2.2 *$F_{\frac{r}{2}}$ is injective on $PP(\delta)$.*

Proof: Assume, in contrary, that there are functions g_1 and g_2 in $PP(\delta)$, $g_1 \neq g_2$ with exactly the same first r Fourier moments. Hence for the difference $g = g_2 - g_1 \neq 0$ we have the vanishing of the first r Fourier coefficients: $f_{\pm j}(g) = 0,\ j = 0, 1, \ldots, \frac{r}{2}$. By Lemma 2.1 we have for the combinatorial complexity of g the bound $\sigma_{PP}(g) \leq r$. By Proposition 2.1 $\sigma_{PP}(g)$ bounds from above the number of sign changes of g. Thus g changes sign at certain points $x_1, \ldots, x_q,\ q \leq r$ (If g flip sign by vanishing on some interval we take only one point -w.l.o.g the middle- in this interval as x_i). W.l.o.g q is an even number (otherwise add $x_{q+1} = 2\pi$).

Let us now construct a trigonometric polynomial with exactly the same sign pattern as g: $Q(x) = \operatorname{sgn}(g(0))(-1)^q\prod_{i=1}^{q}\sin\left(\frac{1}{2}(x-x_i)\right)$. Every $\sin\left(\frac{1}{2}(x-x_i)\right)$ is negative from 0 till x_i and positive from x_i till 2π (this fact is similar to the property of the product suggested in theorem 2.1). Using lemma 2.2 we get the exponential polynomial structure $Q(x)=\sum_{-\frac{q}{2}}^{\frac{q}{2}}\alpha_ne^{-inx}$. Therefore we have $g(x)Q(x) > 0$ everywhere, except the points $x_1, \ldots, x_q$, (and 0 if q was odd at the beginning) and the intervals where $g \equiv 0$ (which are not all the domain since $g \not\equiv 0$). Therefore $\int_{S^1} g(x)Q(x) > 0$. On the other hand, this integral can be expressed as a linear combination of the Fourier coefficients:

$$\begin{aligned} 0 < \int_{S^1} g(x)Q(x) &= \sum_{-\frac{q}{2}}^{\frac{q}{2}}\alpha_n\int_{S^1}e^{-inx}g(x)\,dx\\ &= 2\pi\sum_{-\frac{q}{2}}^{\frac{q}{2}}\alpha_na_n(g)\\ &= 0\end{aligned}$$

since all the Fourier coefficients of g up to $\frac{r}{2}$ vanish by the assumption. This contradiction proves Theorem 2.2.

Remark 1. It is not completely clear, whether we need $r = 2\delta(2\delta+1)$ moments to reconstruct a function from $PP(\delta)$ uniquely. This number r appears as a result of taking the *difference* of two functions in $PP(\delta)$. However, to conclude that a function $g \in PP(\delta)$ vanishes, it is enough the vanishing of its $\delta+1$ first moments or first $\delta+\delta\,(\mathrm{mod}2)$ Fourier moments:

Proposition 2.2 *If for a function $g \in PP(\delta)$ we have $m_i(g) = 0,\ i = 0, 1, \ldots, \delta$ (respectively $a_{\pm i}(g) = 0,\ i = 0, 1, \ldots, \left\lceil \frac{\delta}{2} \right\rceil$), then $g \equiv 0$.*

Proof: Exactly the same as for Theorems 2.1 and 2.2, taking into account that the number of the sigh changes of g is at most δ (notice that here δ does not need to be even as r was in the proof of theorem 2.1).

Remark 2. In Theorem 2.1 (respectively 2.2) we showed that for all sequence of points $\{x_i\}_{i=1}^{q}$ and the basis $\{x^n\}_{n=1}^{\infty}$ $\left(\{e^{inx}\}_{n=-\infty}^{\infty}\right)$ there exists a polynomial (Fourier polynomial) $Q(x)$ of degree q that changes sign at each x_i. It is suggested to expand these theorems to other bases of L^2 as Legendre, Hermite, Laguerre etc. Maybe it is also possible to find criteria on a basis of L^2 s.t it will posses this property. Clearly, in this kind of bases, it is needed first to define generalized polynomials and their degree.

3 Explicit equations for the Fourier inversion.

The injectivity of the Fourier transform on piecewise polynomial functions suggests that the parameters of such a function (i.e. the coordinates of the jump points and the coefficients of the polynomials on each segment) can be uniquely reconstructed from the Fourier data. In this section we construct a system of nonlinear equations which is satisfied by these parameters.

3.1 Fourier transform.

Assume $f : S^1 \to \mathbb{C}$ is a semi-algebraic function s.t $f = \sum_{q=1}^{r} P_q(x)$ where

$$P_q(x) = \sum_{m=0}^{d_q} p_{q,m} x^m \chi_{\Delta_q}(x),$$

$$\chi_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A \end{cases},$$

and the partition is defined

$$\Sigma = \{\Delta_q = [x_q, x_{q+1}] : -\pi = x_1 < x_2 < \ldots < x_r < x_{r+1} = \pi\}.$$

Then

$$a_n(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{2\pi}\sum_{q=1}^{r}\sum_{m=0}^{d_q} p_{q,m}\int_{x_i}^{x_{i+1}} x^m e^{-inx} dx$$

$$= \sum_{m=0}^{d}\sum_{l=0}^{m}\frac{C_{m,l}}{n^{l+1}}\Big\{p_{r,m}x_{r+1}^{m-l}e^{-inx_{r+1}} + \left[\sum_{q=2}^{r}(p_{q-1,m} - p_{q,m})\, x_q^{m-l} e^{-inx_q}\right] - p_{1,m}x_1^{m-l}e^{-inx_1}\Big\}$$

where

$$\begin{aligned} d &= \max\{d_q : q \in \{1, \ldots, r\}\}, & \\ p_{q,m} &= 0 & \text{if } m > d_q, \\ C_{m,l} &= \frac{i^{1-l} m!}{2\pi (m-l)!}. & \end{aligned}$$

This can be written in a compact form using standard inner product

$$a_n(f) = \sum_{m=0}^{d}\sum_{l=0}^{m} C_{m,l}\langle b_m, X_{n,m,l}\rangle \tag{4}$$

where

$$\begin{array}{lll} (b_m)_1 & = -p_{1,m}, & \\ (b_m)_q & = p_{q-1,m} - p_{q,m} & q \in \{2, \ldots, r\}, \\ (b_m)_{r+1} & = p_{r,m}, & \\ (X_{n,m,l})_q & = \frac{x_q^{m-l} e^{inx_q}}{n^{l+1}} & q \in \{1, \ldots, r+1\}. \end{array}$$

Notice that there is a (simple) unique linear transformation between $(b_m)_q$ and the non zero $p_{q,m}$. Using the Lambert W function $\left(z = W(z)\, e^{W(z)}\right)$ we can find that for $m \neq l$

$$\begin{array}{ll} (X_{n,m,m})_q = \frac{x_q^{m-l} e^{inx_q}}{n^{l+1}} & \Longleftrightarrow \\ \\ n^{l+1} (X_{n,m,l})_q = \left(\frac{in}{in}\right)^{m-l} x_q^{m-l} e^{inx_q} & \Longleftrightarrow \\ \\ i^{m-l} n^{m+1} (X_{n,m,l})_q = (inx_q)^{m-l} e^{inx_q} & \Longleftrightarrow \\ \\ \frac{in^{\frac{m+1}{m-l}}}{m-l} (X_{n,m,l})_q^{\frac{1}{m-l}} = \frac{inx_q}{m-l} e^{\frac{inx_q}{m-l}} & \Longleftrightarrow \\ \\ x_q = \frac{m-l}{in} W\left(\frac{in^{\frac{m+1}{m-l}}}{m-l} (X_{n,m,l})_q^{\frac{1}{m-l}}\right) & . \end{array}$$

Using the fact that $x_q \in [-\pi, \pi]$ we can choose the right branch of W (For more details on Lambert function see [4–6]).

For $m = l$ it is easy to see that:

$$x_q = -\frac{i}{n} \ln\left(n^{m+1} (X_{n,m,l})_q\right)$$

Clearly, fixing q, there is a strong algebraic relation between the $(X_{n,m,l})_q$'s . In the case of piecewise constant function we get $d = 0$ hence

$$na_n(f) = \frac{i}{2\pi} \langle b, X_n \rangle$$

where

$$\begin{array}{lll} b_1 & = -p_1 & \\ b_q & = p_{q-1} - p_q & q \in \{2, \ldots, r\} \\ b_{r+1} & = p_r & \\ (X_n)_q & = e^{inx_q} & q \in \{1, \ldots, r+1\} \end{array}$$

(This last set of equations will rise as well from the target detection problem, noting that $e^{inx_q} = (e^{ix_q})^n$)

3.1.1 Another approach to the non linear equations of the Fourier transform.

Consider the case in which all we know about our function f is that $f \in PP(\delta)$. If f is a polynomial of degree δ we will need to find $\delta+1$ coefficients of f, if f is a piecewise constant function on $[0,2\pi]$ we will need to find δ partitioning points in $[0,2\pi]$. We need to think of f as a piecewise polynomial function (each polynomial of degree less then or equal to δ) over a partition of $[0,2\pi]$ with δ points. Thus we need to find $(\delta+1)^2+\delta = O(\delta^2)$ (as $r = \delta(\delta+1) = O(\delta^2)$) parameters to define f (possibly some of the leading coefficients will be zero or some of the partition points will equal their neighbor). Using the previous notation we get that if $f \in PP(\delta)$ then $d_1 = \ldots = d_r = r = \delta$ hence

$$a_n(f) = \sum_{m=0}^{\delta}\sum_{l=0}^{m} C_{m,l}\langle b_m, X_{n,m,l}\rangle .$$

3.2 Moment transform.

Using the same notation in 3.1 we compute the moment transform of f

$$\begin{aligned}
&m_n(f) = \int_{-\pi}^{\pi} f(x)\, x^n dx = \textstyle\sum_{q=1}^{r}\sum_{m=0}^{d_q} p_{q,m}\int_{x_i}^{x_{i+1}} x^{m+n}dx \\
&= \textstyle\sum_{m=0}^{d} \frac{1}{n+m+1}\left[p_{r,m}x_{r+1}^{n+m+1} + \sum_{q=2}^{r}(p_{q-1,m}-p_{q,m})\, x_q^{n+m+1} - p_{1,m}x_1^{n+m+1}\right] \\
&= \textstyle\sum_{m=0}^{d} \frac{1}{n+m+1}\langle b_m, X_{n,m}\rangle
\end{aligned}$$

where

$$(X_{n,m})_q = x_q^{n+m+1} \quad q \in \{1,\ldots,r+1\}$$

and again for piecewise constant function we get $d=0$ hence

$$(n+1)\, m_n(f) = \langle b, X_n\rangle$$

where

$$(X_n)_q = x_q^{n+1} \quad q \in \{1,\ldots,r+1\} \ .$$

(This set of equations, as we mentioned, will rise as well from the target detection problem)

3.2.1 Another approach to the non linear equations of the Moment transform.

Again using the same logic of section 3.1.1 we get

$$m_n(f) = \sum_{m=0}^{\delta} \frac{1}{n+m+1} \langle b_m, X_{n,m} \rangle . \tag{5}$$

We get systems of exactly the same structure!

In each case we have to try to *solve these systems explicitly.* As mentioned before we shall not address here the general case. One important special case is treated in section §4.7

4 The Colliding Targets Detection problem (CTD).

4.1 The general manifestation of CTD.

Assume we have N targets in the space and we transmit from a radar at the origin a signal M through the space. The signal reflects from the targets and return with different amplitudes and phases shifts back to our radar where we can measure the superposition reflected signals from the targets with some probabilistic/physical noise added on the way. If our transmitted signal is $M(t)$, the positions of the targets are $\{\tau_i\}_{i=1}^{N}$ and the different (complex) amplitudes are $\{A_i\}_{i=1}^{N}$ then the reflected signal will be

$$F(t) = \sum_{n=1}^{N} A_n M(t - \tau_n) + \epsilon(t). \tag{1}$$

Here $\epsilon(t)$ is the added noise. We would like to find from the reflected signal the parameters N, A_n and τ_n.

4.2 Special cases of Target Detection problem.

We approach the general target detection problem step by step, starting with some simplified but still very instructive special cases.

4.2.1 Simplified Target Detection problem.

Here we assume that the reflected radar signal is the sum of the original pulse, reflected from different point targets. We assume in addition that the amplitude and the initial phase of the reflected pulses are preserved. This last assumption is not realistic, and we remove it in the next section.

Under the above assumptions the only parameters of F are the targets positions, i.e. the time shifts of the reflected pulses.

We investigate this problem on the base of the “correlation function” which significantly reduces the noise.

4.2.2 Extended Target Detection problem.

As above, we assume that the reflected radar signal is the sum of the original pulse, reflected from different point targets. However, as it happens in reality, the amplitude and the initial phase of the reflected pulses may vary, and they are not known a priori.

We get more free parameters than in the simplified problem above. We estimate these parameters on the base of a more careful analysis of the correlation function.

4.2.3 The problem of detection of colliding targets CTD.

The problem of an accurate detection of two or more targets roughly at the same distance is practically very important. The resolution of the conventional methods drops in such situations. We investigate this problem combining the analysis of the convolution function with the description of the typical patterns of this function.

A reasonable assumption will be that $M(t)$ is supported on a finite domain let say $[-a, a]$. Therefore if $m = \min \{|\tau_i - \tau_j| : i \neq j\} > 2a$, it is easy to detect the position of the targets from the maximal points of the correlation function of F with M. This is for the case when the noise $|\epsilon| << |F|, |M|$. If the noise ϵ (as in 1) is to large, it is possible to lower it with the convolution mask - will be explained. In the case that $m < 2a$ we fall below the ordinary resolution of the system and we need to further analyze the reflected signal to identify the parameters we look for.

4.3 The starting point - the case of one target.

In the case of one target the fact that the amplitude of the reflected signal is not a priori known is not essential - we can always normalize the signal.

In this paper we always start with the convolution function: given a reflected signal

$$F(t) = AM(t - \tau) + \epsilon(t), \tag{2}$$

where A is the signal amplitude, $M(t)$ the standard radar pulse (reflected) and $\epsilon(t)$ the noise, we construct the *convolution function*

$$Cf(t) = F * M = AM * M + \epsilon * M. \tag{3}$$

There are at least two reasons to use the convolution function as the input: first, its Sound to Noise Ratio (SNR) is usually much better than for the original signal F. Secondly, this function is efficiently and robustly computed in the actually working detection algorithms, so it presents a natural input for any further processing. Notice, however, that *it is not necessary to use the convolution with exactly the radar pulse M*. Other convolution kernels can be used. Another aspect is that if our transmitted signal M is not smooth enough it is possible to gain a smooth convolution mask after convolving with smooth enough function.

Still, the maximum of the convolution function can be strongly shifted by the noise. Indeed, this maximum is a *point-wise, and not an integral* characteristic of the convolution. Accordingly, with a high probability it can deviate from the actual target position. In order to further improve the accuracy of the detection we repeat the convolution step. Indeed, denoting the "convolution mask" $M * M$ by M_1 , up to now we have

$$CF\left(t\right) = F * M = A\left(M * M\right) + \epsilon * M = A \cdot M_1\left(t - \tau\right) + \epsilon_1\left(t\right), \quad (4)$$

where the convolved noise $\epsilon_1\left(t\right)$ is usually much smaller than the original one (For some numerical estimations see §4.8.3). The equation 4 has exactly the same form as the original equation 2 and we can apply the second convolution step:

$$C_2F\left(t\right) = CF * M_1 = A\left(M_1 * M_1\right) + \epsilon_1 * M_1. \quad (5)$$

(again the convolved kernel can be arbitrary and not M_1) The expected twice convolved noise $\epsilon_2\left(t\right)$ is usually much smaller than $\epsilon_1\left(t\right)$, and we take as the estimated target position the maximum of the second convolution $C_2F\left(t\right)$.

4.4 Two targets with the same reflected amplitudes.

In this case the reflected signal has the form

$$F\left(t\right) = A[M\left(t - t_1\right) + M\left(t - t_2\right)] + \epsilon\left(t\right), \quad (6)$$

where A is the signal amplitude. Accordingly, the convolution function has the form

$$CF\left(t\right) = F * M = A[M\left(t - t_1\right) + M\left(t - t_2\right)] * M + \epsilon * M. \quad (7)$$

Hence it is the sum of the shifted convolution masks and the convolved noise

$$CF(t) = A[M_1(t - t_1) + M_1(t - t_2)] + \epsilon_1(t)\,. \tag{8}$$

Denoting by $M_2 = M_1 * M_1$ the second convolution mask of the radar pulse, we get for the second convolution of the signal

$$C_2F(t) = CF * M_1 = A[M_2(t - t_1) + M_2(t - t_2)] + \epsilon_2(t)\,. \tag{9}$$

The same form of the signal is preserved also for the higher iterated convolutions.

4.4.1 Identifying the middle point.

Another reasonable assumption is that the transmitted signal is symmetric with respect to the middle point $t_0 = \frac{1}{2}(t_1 + t_2)$.

Let us assume that the distance $t_2 - t_1$ between the two targets is small. In this case the second convolution of the signal $C_2F(t)$ has a unique well defined extremum. In our case, in the absence of the noise this extremum is exactly at the point $t_0 = \frac{1}{2}(t_1 + t_2)$, because of the symmetry of the signal. In the presence of the noise we take this extremum as the estimate of this middle point. Alternatively, we can estimate the position of this extremum using higher iterated convolutions.

4.4.2 Identifying the time shift.

It remains to find the shift $2\tau = t_2 - t_1$. To simplify the notations, let us assume that $t_0 = \frac{1}{2}(t_1 + t_2) = 0$. We can naturally separate two steps in our algorithm:

The first step is forming the iterated convolutions, in order to reduce the noise. At present we work with the second convolution, but in principle one can iterate the convolution step more times. It is a separate question of what number of convolution is optimal. Although each convolution step reduces the noise, it has also a "low-pass filter" effect on the signal itself, which ultimately reduces the resolution.

The second step is identifying the time shift parameter τ in the signal of the form

$$F(t) = A[G(t - \tau) + G(t + \tau)] + \epsilon(t) \tag{10}$$

where $G(t) = M(t - t_0)$. Indeed, the iterated convolutions preserve this form of a signal. So we describe below this second step, assuming that the noise $\epsilon(t)$ is sufficiently low.

4.4.3 First and second order expansion of the signal.

In accordance to what has been explained above, let us assume that the measured reflected signal $F(t)$ is given by

$$F(t) = A[G(t-\tau) + G(t+\tau)] + \epsilon(t), \tag{11}$$

where $G(t)$ is an a priori known twice differentiable function. Assuming, as above, that τ is small enough, we get in the first approximation

$$G(t-\tau) + G(t+\tau) \approx 2G(t) + G'(t)\tau - G'(t)\tau = 2G(t). \tag{12}$$

We see that because of the cancellation of the first order terms in τ the measured reflected signal $F(t) = A[G(t-\tau) + G(t+\tau)] + \epsilon(t)$ *does not depend on τ in the first approximation*. This fact presumably explains some of the difficulties in the CTD.

Let us write the second order expansion in τ:

$$G(t-\tau) + G(t+\tau) \approx 2G(t) + G''(t)\tau^2, \tag{13}$$

or

$$F(t) \approx 2A \cdot G(t) + 2A \cdot G''(t)\tau^2 + \epsilon(t). \tag{14}$$

Assuming that the (integrated) noise is small with respect to τ^2, the "functional" equation 14 allows us to produce a system of algebraic equations in order to estimate both the time shift τ and the amplitude A.

4.4.4 Obtaining and solving equations on A and τ^2.

In order to produce from the functional relation 14 robust *scalar* equations on A and τ^2, we integrate 14 with different weights. Let us fix first a function $U(t)$ orthogonal to $G(t)$ (notice that $G(t)$ is an a priori known function). We get

$$\int U(t)F(t) \approx 2A\left(\int U(t)G''(t)\right)\tau^2 + \int U(t)\epsilon(t). \tag{15}$$

This is the first of our equations. To produce the second equation, let us fix a function $W(t)$ orthogonal to $G''(t)$. Integrating, we obtain

$$\int W(t)\, F(t) \approx 2A \int W(t)\, G(t) + \int W(t)\, \epsilon(t)\,. \tag{16}$$

Dropping the noise, we conclude from this last equation that

$$A \approx \frac{\int W(t)\, F(t)}{2 \int W(t)\, G(t)}. \tag{17}$$

Substituting this expression for A into 15 we get

$$\tau^2 \approx \frac{\int U(t)\, F(t) \int W(t)\, G(t)}{\int U(t)\, G''(t) \int W(t)\, F(t)}. \tag{18}$$

The expressions 17 and 18 provide a complete answer to our problem, assuming that the integrated noise $\int U(t)\, \epsilon(t)$ is small with respect to τ^2.

Remark. One can use other approaches to the analysis of the equation 14:

1. If the amplitude A is known and if we believe that *the point-wise* noise level is low with respect to the expected value of τ^2, we just substitute $t = 0$ into the equation 14. We get

$$\tau^2 \approx \frac{F(0) - 2A \cdot G(0)}{2A \cdot G''(0)}. \tag{19}$$

2. If the point-wise noise is still too high, we can try to "integrate it out" convolving the equation 14 with a certain known kernel K, not necessarily U or W used above:

$$\int K(t)\, F(t) \approx 2A \int K(t) \left[G(t) + G''(t)\, \tau^2\right] + \int K(t)\, \epsilon(t)\,.$$

Assuming that A is known, this leads to the following estimate for τ^2:

$$\tau^2 \approx \frac{\int K(t)\, F(t) - 2A \int K(t)\, G(t)}{2A \int K(t)\, G''(t)}. \tag{20}$$

More generally, using two kernels K_1 and K_2 we get a system of two equations on A and τ^2.

3. Notice that for any exponential (or polynomial of degree < 2) function g (also complex i.e in particular trigonometric functions) and for any kernel U s.t $\int gU = 0$ we will get also that $\int g''U = 0$ (since $g'' \sim g$), so this method could not work for signal s.t its second convolution mask is exponential. Recall that G is our second convolution mask of the transmitted signal M i.e

$$G(t) = ((M * M) * (M * M))(t)$$

clearly if M is exponential so is G, and if G is exponential function then by invoking the convolution theorem twice so is M i.e this method could not work for exponential signals (in particular trigonometric ones). On the other hand since we assume our signal is supported on a finite interval this kind of a function is not of our concern.

4. For f, g functions s.t $g \not\equiv 0$ define $f \ominus g = f - g\frac{\int f(t)g(t)}{\int g^2(t)}$. Let us assume that

$$\int G^2(t) \int G''^2(t) \neq \left(\int G''^2(t)\, G(t)\right)^2$$

and

$$\int G(t)\, F(t) \int G''^2(t) \neq \int G''(t)\, F(t) \int G(t)\, G''(t)$$

then we can suggest a universal kernels U and W in the form:$U = G'' \ominus G$ and $W = G \ominus G''$ using them we get

$$\begin{array}{ll} A & = \frac{1}{2}\frac{\int G(t)F(t)\int G''^2(t) - \int G''(t)F(t)\int G''(t)G(t)}{\int G^2(t)\int G''^2(t) - \left(\int G''(t)G(t)\right)^2} \\ \tau^2 & = 2\frac{\int F(t)G''(t)\int G^2(t) - \int F(t)G(t)\int G''(t)G(t)}{\int F(t)G(t)\int G''^2(t) - \int F(t)G''(t)\int G''(t)G(t)} \end{array}. \tag{21}$$

See section §4.8.4 for this solution.

4.5 Two targets with unknown reflected amplitudes.

In this section we assume that the measured reflected signal $F(t)$ is given by

$$F(t) = A_1 G(t - t_1) + A_2 G(t - t_2) + \epsilon(t), \tag{22}$$

where $G(t)$ is an a priori known twice differentiable function. Using the same methods as above we can estimate a certain "common amplitude" of the signal, and thus "normalize" the expression for F in such a way that only one unknown amplitude parameter appears. To simplify the computations below, let us assume that $A_1 = 1$ and so $F(t) = G(t - t_1) + A \cdot G(t - t_2) + \epsilon(t)$.

As the first step we look for the extremum t_0 of F. (Let us remind that F is assumed to be a certain iterated convolution of the original measured reflected signal, so the noise ϵ is assumed to be small). Accordingly, we suppose that we have an accurate practical method for the estimation of t_0. Now let us express t_0 analytically.

4.5.1 The expected extremum of the signal.

As above, the time shift $\tau = t_2 - t_1$ between the two targets is assumed to be small. Using the second order approximation $G(t) \approx G(0) + \frac{1}{2}G''(0)\, t^2$ (where 0 is assumed to be the extremum and the center of G) we find that $F(t) \approx G(0) + \frac{1}{2}G''(0)(t - t_1)^2 + A[G(0) + \frac{1}{2}G''(0)(t - t_2)^2]$, which is equal to $Const + \frac{1}{2}(1 + A)\, G''(0)\, [(t - t_1)^2 + A(t - t_2)^2]$. The extremum of the last function is at the point $t_0 = \frac{t_1 + At_2}{1+A}$. Notice that $t_0 - t_1 = \frac{A}{1+A}(t_2 - t_1) = \frac{A}{1+A}\tau,\ t_2 - t_0 = \frac{1}{1+A}(t_2 - t_1) = \frac{1}{1+A}\tau$.

Denoting $\frac{A}{1+A}$ by α and $\frac{1}{1+A}$ by $\beta,\ \alpha + \beta = 1$, and shifting the origin into the point t_0, we can write

$$F(t) = G(t + \alpha\tau) + A \cdot G(t - \beta\tau) + \epsilon(t)\,. \tag{23}$$

Remind that the extremum point t_0 can be practically identified in a robust way, and hence the time shift as above can be performed explicitly.

4.5.2 First and second order expansion of the signal.

Assuming, as above, that τ is small enough, we get in the first approximation

$$F(t) \approx (1 + A)\, G(t) + (\alpha - A\beta)\, G'(t)\, \tau = (1 + A)\, G(t)\,, \tag{24}$$

since $\alpha - A\beta = \frac{A}{1+A} - \frac{A}{1+A} = 0$. Thus, also for nonequal amplitudes the first order approximation in τ vanishes. Notice that for $A = 1$ we get the case of equal amplitudes, considered in section 4.4 above.

So let us write the second order expansion: using $\alpha = (A\beta^2 + \alpha^2)$

$$F(t) \approx (1 + A)\, G(t) + \frac{1}{2}\alpha G''(t)\, \tau^2 + \epsilon(t)\,. \tag{25}$$

In this expression we have two unknowns: A and τ (α is being expressed through A).

4.5.3 Obtaining and solving equations on A and τ.

Integrating the equation 25 with different weights, in the same way as in section §4.4 above, we can extract from it some simple algebraic relations between A and τ. First, let us fix a function $U(t)$ orthogonal to $G(t)$ (remind that $G(t)$ is a known function). Integrating 25 with $U(t)$ we get

$$\int U(t)\,F(t) \approx \frac{1}{2}\alpha\left(\int U(t)\,G''(t)\right)\tau^2 + \int U(t)\,\epsilon(t)\,, \tag{26}$$

since $\int U(t)\,G(t) = 0$ by the choice of $U(t)$. Assuming that the integrated noise is small, we get from here that

$$\alpha\tau^2 \approx 2\frac{\int U(t)\,F(t)}{\int U(t)\,G''(t)}. \tag{27}$$

The right hand side of this equation we can compute directly from the measured signal $F(t)$. The equation 27 is the first in our basic nonlinear system of equations for the unknowns A and τ.

Now let us integrate the equation 25 with a different weight $W(t)$, not orthogonal to $G(t)$ but orthogonal to $G''(t)$. We obtain, dropping the noise

$$\int W(t)\,F(t) \approx (1+A)\int W(t)\,G(t)\,. \tag{28}$$

The equation 28 is the second one in our basic nonlinear system of equations for the unknowns A and τ. Now we can easily solve the system of equations 27 and 28: Solving 28 we obtain

$$A \approx \frac{\int W(t)\,F(t)}{\int W(t)\,G(t)} - 1. \tag{29}$$

Substituting 28 into 27, in particular, expressing through A the coefficient α, we finally find τ^2

$$\tau^2 \approx 2\frac{\int U(t)\,F(t)\int W(t)\,F(t)}{\int U(t)\,G''(t)\left[\int W(t)\,F(t) - \int W(t)\,G(t)\right]}.$$

See section §4.8.5 for this solution.

Remark 1. We can take different orthogonal and not orthogonal to G kernels and integrate them with the equation 25. In particular, we can take as a kernel G itself. In this way we get different equations of the form 27 and 28. Presumably, using a few of such equations may improve the final accuracy of our estimates. As well we can take the universal kernels we mentioned in §4.5.3 then we get:

$$\int U(t)\,F(t) = \int G''(t)\,F(t) - \frac{\int G(t)\,F(t)\int G(t)\,G''(t)}{\int G^2(t)}$$

$$\int W(t)\,F(t) = \int G(t)\,F(t) - \frac{\int G''(t)\,F(t)\int G''(t)\,G(t)}{\int G''^2(t)}$$

$$\int U(t)\,G''(t) = \int G''^2(t) - \frac{\left(\int G(t)\,G''(t)\right)^2}{\int G^2(t)}$$

$$\int W(t)\,G(t) = \int G^2(t) - \frac{\left(\int G(t)\,G''(t)\right)^2}{\int G''^2(t)}$$

as long the appropriate integrals does not vanish.

4.6 A General CTD Problem.

4.6.1 Generating the set of equations to solve CTD.

Let us assume that our transmitted signal (or its second convolution) M is in $C^{l+1}(\mathbb{R})$ and has compact support. *Assume that we know that there are N targets all of them near the origin* and that the noise is small. Our measured signal now is

$$F(t) = \sum_{n=1}^{N} A_n M(t-\tau_n) + \epsilon(t). \tag{30}$$

Let us develop $F(t)-\epsilon(t)$ in Taylor expansion up to the $l^{\underline{th}}$ derivative with the remainder in Lagrange form:

$$F(t)-\epsilon(t) = \sum_{k=0}^{l}\frac{1}{k!}\sum_{n=1}^{N} A_n M^{(k)}(t)\,\tau_n^k + \frac{1}{(l+1)!}\sum_{n=1}^{N} A_n M^{(l+1)}(t+c_n)\,\tau_n^{l+1}.$$

Denoting

$$\rho_N(t) = \frac{1}{(l+1)!}\sum_{n=1}^{N} A_n M^{(l+1)}(t+c_n)\,\tau_n^{l+1} \tag{31}$$

gives us

$$F(t) - \epsilon(t) - \rho_N(t) = \sum_{k=0}^{l} M^{(k)}(t)\,\beta_k \tag{32}$$

where $\beta_k = \sum_{n=1}^{N} \frac{1}{k!} A_n \tau_n^k$.

Now we shall address a theoretic problem. The result we shall use to analyze CTD.

4.6.2 Finding the Dual basis for a function and its derivatives.

Consider the space $L^2(\mathbb{R})$ and let $f \in C^{l+1}(\mathbb{R})$ be a function with a compact support s.t for all $n \in \mathbb{N}$ the set $\left\{f^{(i)}\right\}_{i=0}^{n}$ is linearly independent (as long the derivatives are defined). In other words f is not a solution of an o.d.e with constant coefficients. (equivalently f is not a sum of polynomials multiplied by exponents, in particular any function with compact support).
For fix $n \le l$, we would like to find a set of functions $\{U_i\}_{i=0}^{n}$ s.t $\left\langle f^{(i)}, U_j\right\rangle = \delta_{i,j}$ in the L^2 inner product.
We would like to find the dual basis of $\left\{f^{(i)}\right\}_{i=0}^{n}$ in the subspace $W_n = \mathrm{sp}\left\{f^{(i)}\right\}_{i=0}^{n}$. Let us define $G_n(f)_{i,j} = \left\langle f^{(i)}, f^{(j)}\right\rangle$ the $n+1$ by $n+1$ Gramm-Schmidt matrix related to f (which is invertible since the set $\left\{f^{(i)}\right\}_{i=0}^{n}$ is linearly independent).

Proposition 4.1 *$U_j = \sum_{k=0}^{n} a_{j,k} f^{(k)}$ is the dual basis of $\left\{f^{(i)}\right\}_{i=0}^{n}$ in W_n iff $a_{i,j} = \left(G_n(f)^{-1}\right)_{i,j}$*

Proof:
<u>"$\Leftarrow$"</u>: Let us look at the inner product

$$\left\langle U_i, f^{(j)}\right\rangle = \sum_{k=0}^{n} a_{i,k}\left\langle f^{(k)}, f^{(j)}\right\rangle = \sum_{k=0}^{n}\left(G_n(f)^{-1}\right)_{i,k} G_n(f)_{k,j} = Id_{i,j} = \delta_{i,j}.$$

<u>"$\Rightarrow$"</u>: We shall show that in W_n there is only one dual basis to $\left\{f^{(i)}\right\}_{i=0}^{n}$ and by that prove this implication. Assume that $\left\{\tilde{U}_i\right\}_{i=0}^{m} \subset W_n$ is a dual basis, i.e for all

$i, j = \{0, \dots, n\}$ $\left\langle \tilde{U}_i, f^{(j)} \right\rangle = \delta_{i,j}$. Therefor

$$\left\langle \tilde{U}_i - U_i, f^{(j)} \right\rangle = \left\langle \tilde{U}_i, f^{(j)} \right\rangle - \left\langle U_i, f^{(j)} \right\rangle = \delta_{i,j} - \delta_{i,j} = 0$$

from here we conclude that $\tilde{U}_i - U_i \in W_n^{\perp}$. Since $\tilde{U}_i, U_i \in W_n$ we get that $\tilde{U}_i - U_i \in W_n \cap W_n^{\perp} = \{0\} \Rightarrow \tilde{U}_i = U_i$. And by this we proved this Proposition.

Finding this we can easily see that if we take $v_i \in W_n^{\perp}$ and define $\hat{U}_i = U_i + v_i$ then we still have $\left\langle \hat{U}_i, f^{(j)} \right\rangle = \delta_{i,j}$. From the uniqueness of U_i in W_n follows that this is all the freedom we can have in the choice of $\hat{U}_i$.

Proposition 4.2 *Given V_n s.t $\langle V_n, f^{(i)} \rangle = \delta_{n,i}$ for all $i \in \{0, \dots, n, \dots, 2n\}$ the relation $V_{j-1} = -V_j'$ generate a dual linear independent set for $\left\{ f^{(i)} \right\}_{i=0}^{n}$ in W_{2n}.*

Proof: Assume we have found $V_n, V_{n-1}, \dots, V_j$ s.t for all $i \in \{n, n-1, \dots, j\}$ and $r \in \{1, \dots, n, \dots, n+j\}$

$$\left\langle V_i, f^{(r)} \right\rangle = \delta_{i,r}$$

then for all $r \in \{1, \dots, n, \dots, n+j-1\}$

$$\left\langle V_{j-1}, f^{(r)} \right\rangle = \int_{\mathbb{R}} V_{j-1} f^{(r)} = \int_{\mathbb{R}} \left(-V_j'\right) f^{(r)}$$

Using integration by parts and the fact that the support of f is compact we get

$$\left\langle V_{j-1}, f^{(r)} \right\rangle = \int_{\mathbb{R}} V_j f^{(r+1)} = \delta_{j,r+1} = \delta_{j-1,r}.$$

By this induction we proved the duality property of $\{V_j\}_{j=0}^{n}$ w.r.t $\left\{ f^{(i)} \right\}_{i=0}^{n}$. Assume that there exists $\{a_j\}_{j=0}^{n}$ constants s.t $\sum_{j=0}^{n} a_j V_j = 0$ then for all $i \in \{0, \dots, n\}$

$$0 = \left\langle 0, f^{(i)} \right\rangle = \left\langle \sum_{j=0}^{n} a_j V_j, f^{(i)} \right\rangle = \sum_{j=0}^{n} a_j \left\langle V_j, f^{(i)} \right\rangle = \sum_{j=0}^{n} a_j \delta_{i,j} = a_i.$$

Therefor $\{V_j\}_{j=0}^{n}$ is a linear independent set. By this we proved the Proposition.

Proposition 4.3 *$V_n = \sum_{i=0}^{2n} b_i f^{(i)}$ where $b = G_{2n}(f)^{-1} e_n$ and $(e_i)_j = \delta_{i,j}$ for all $i, j \in \{0, \dots, 2n\}$.*

Proof:

$$\begin{aligned} b = G_{2n}(f)^{-1} e_n &\iff G_{2n}(f)\, b = e_n \\ &\iff (G_{2n}(f)\, b)_i = (e_n)_i = \delta_{i,n} \\ &\iff \textstyle\sum_{j=0}^{2n} (G_{2n}(f))_{i,j}\, b_j = \delta_{i,n} \\ &\iff \textstyle\sum_{j=0}^{2n} \left\langle f^{(j)}, f^{(i)} \right\rangle b_j = \delta_{i,n} \\ &\iff \left\langle \textstyle\sum_{j=0}^{2n} b_j f^{(j)}, f^{(i)} \right\rangle = \delta_{i,n} \\ &\iff \left\langle V_n, f^{(i)} \right\rangle = \delta_{i,n} \end{aligned}$$

Remark: Notice that in order to find all the $\{U_i\}_{i=0}^{n}$ together as in Proposition 4.1, we made all the calculation in space of dimension $n+1$. To find V_n and to develop the rest by differentiating as in Proposition 4.2, we made calculations in space of dimension $2n+1$.

Proposition 4.4 *If $g \in C_0(\mathbb{R})$ is twice differential function then $\int_{\mathbb{R}} gg'' = 0 \iff g \equiv 0$*

Proof: The proof is easy using integration by parts and zero value on the boundaries for g and g'.
Conclusion: $\left\{f^{(i)}\right\}$ (respectivly $\{V_i\}$ from proposition 4.2) is not an orthogonal set.
Remark: A suggested question for future work can be :
How close to orthogonal we can get it?

4.6.3 Producing the explicit system of equations

Assume more that $\left\{M^{(i)}\right\}_{i=0}^{l}$ is a linear independent set (since the support of M is compact it is a reasonable assumption). Convolving 32 with the kernels U_i we just found we get

$$\langle F, U_i\rangle - \langle \epsilon, U_i\rangle - \langle \rho_N, U_i\rangle = \sum_{k=0}^{l} \left\langle M^{(k)}(t), U_i(t) \right\rangle \beta_k = \sum_{k=0}^{l} \delta_{k,i}\beta_k = \beta_i$$

that is

$$\langle F, U_i\rangle - \langle \epsilon, U_i\rangle - \langle \rho_N, U_i\rangle = \beta_i \tag{33}$$

In equation 33 the expression $\langle F, U_i \rangle$ is the actual "measurement". The noise $\langle \epsilon, U_i \rangle$ is unknown, while for the remainder term we have the following bound (using Cauchy-Schwartz inequality)

$$
\begin{aligned}
|\langle \rho_N, U_i \rangle| &= \left| \frac{1}{(l+1)!} \sum_{n=1}^{N} A_n \left\langle U_i, M^{(l+1)}\left(t + c_n\right) \right\rangle \tau_n^{l+1} \right| \\
&\leq \frac{N}{(l+1)!} R_A R_\tau^{l+1} \|M^{(l+1)}\| \cdot \|U_i\| =: R_{N,i} \qquad (34)
\end{aligned}
$$

where R_A and R_τ are a-priori bounds for the amplitudes and the positions of the targets.

Side remarks:

1. R_A can be taken as the initial amplitude of the transmitted signal. The energy conservation law explains why it is a bound (maybe not tight enough).
2. R_τ can be taken as a from §4.2.3 or smaller if we investigate more.

Assuming the noise is much smaller than the remainder we can omit the correspondence term to get

$$
|\langle F, U_i \rangle - \beta_i| \leq R_{N,i}. \qquad (35)
$$

Our basic assumption is that R_τ is small. Taking it to the power of $l+1$ gives us even smaller parameter. Taking this assumption under account we can use the Lagrange reminder as a good bound for estimation.

Now we finally obtain the set of equations

$$
\gamma_k := k! \langle F, U_k \rangle = \sum_{n=1}^{N} A_n \tau_n^k \qquad (36)
$$

where the A_n's and τ_n's are the unknowns.
The number N is not easy to find, this methods assumes we know the total number of targets a-priori.

4.7 Solution Methods of 36.

We shall introduce 2 methods to solve 36:

1. Analytic method suggested by Kisunko in [7].
2. Special case for 2 colliding targets.

4.7.1 Analytic method suggested by Kisunko in [7].

Let us find a Generating function for the sequence γ_k's (as in 36). Let z be an auxiliary coefficient. Using it we get

$$g(z) = \sum_{k=0}^{\infty} \gamma_k z^k = \sum_{k=0}^{\infty} \sum_{n=1}^{N} A_n \tau_n^k z^k = \sum_{n=1}^{N} A_n \sum_{k=0}^{\infty} (z\tau_n)^k = \sum_{n=1}^{N} \frac{A_n}{1 - z\tau_n}. \tag{37}$$

Hence g is a rational function of degree N.
From 37 we can see that τ_n^{-1} are the poles of g and $A_n = -\tau_n \mathrm{Res}\left(g, \tau_n^{-1}\right)$.
Notice that the γ_k's are the Taylor coefficients of g for $k \in \{0, 1, \ldots\}$.

Let us remind that the Taylor coefficients of a rational function of degree N satisfy a linear recurrence relation of length $2N$.
So our problem is to find explicitly the function g from the first $2N$ coefficients γ_k's.

Proposition 4.5 *Let $h(x) = \frac{p(x)}{q(x)}$ be a rational function s.t $q(x_0) \neq 0$ and $\deg p < \deg q = n$. Then there exists $\{c_i\}_{i=0}^{n-1}$ s.t*

$$a_{k+n} = \sum_{i=0}^{n-1} c_i a_{k+i}$$

where a_i are the Taylor coefficients of h around x_0.

Proof: Let us differentiate $p(x), n+k$ times. Since $\deg p < n \leq n+k$ we get

$$0 = \frac{d^k}{dx^k} 0 = \frac{d^k}{dx^k} \frac{d^n}{dx^n} p(x) = \frac{d^{n+k}}{dx^{n+k}} p(x) = \frac{d^{n+k}}{dx^{n+k}} (h(x) q(x)).$$

Therefore

$$0 = \frac{d^{n+k}}{dx^{n+k}} (h(x) q(x)) = \sum_{j=0}^{n+k} \binom{n+k}{j} h^{(n+k-j)}(x) q^{(j)}(x) = \sum_{j=0}^{n} \binom{n+k}{j} h^{(n+k-j)}(x) q^{(j)}(x).$$

Evaluating at $x = x_0$ and rearranging we get

$$0 = \sum_{j=0}^{n} \frac{(n+k)!}{j!} \frac{h^{(n+k-j)}(x_0)}{(n+k-j)!} q^{(j)}(x_0) = \sum_{j=0}^{n} \frac{q^{(j)}(x_0)(n+k)!}{j!} a_{n+k-j}.$$

Using the fact that $q(x_0) \neq 0$ we get

$$a_{k+n} = \sum_{j=1}^{n} \frac{-q^{(j)}(x_0)}{q(x_0)\, j!} a_{n+k-j} = \sum_{i=0}^{n-1} \frac{-q^{(n-i)}(x_0)}{q(x_0)(n-i)!} a_{k+i} =: \sum_{i=0}^{n-1} c_i a_{k+i}.$$

By this we proved the Proposition.

Assume we have the first $2N$ equations of 36. In order to find g we have to preform the following two steps:

1. Finding the linear coefficients of the recursion relation between the first $2N$ $\gamma_k's$ (for the solvability of this linear system see [7]). By this step we find the denominator of g and therefore the τ_n's.
2. Finding the nominator of g and through it the A_n's.

Notice that in our case, if q is the denominator of g then $q(0) = 1 \neq 0$. Having this and the c_i's we can construct q (as in 37) from its derivatives at 0. In this case

$$c_i = -\frac{q^{N-i}(0)}{(N-i)!}. \tag{38}$$

Therefore

$$q(z) = -\sum_{i=0}^{N} c_{N-i} z^i. \tag{39}$$

Let us write $g(z) = \frac{p(z)}{q(z)}$ where q is known. p is just a polynomial of degree less than N. Equating the first $N_1\,\gamma_k$'s to the first $N+1$ Taylor coefficients of g gives us $N+1$ linear equations to solve for p coefficients. (for details see [7, 8])

4.7.2 Special cases for 2 colliding targets.

Notice that if the targets are symmetrical around zero then

$$F(t) = A_1 M(t+\tau) + A_2 M(t-\tau) \tag{40}$$

and its Taylor expansion is

$$F(t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(A_1 + (-1)^n A_2\right) M^{(n)}(t)\, \tau^n. \tag{41}$$

Integrating against the dual kernels w.r.t $\left\{M^{(i)}\right\}$ (see §4.6.2) we get

$$|\langle F, U_m \rangle - \frac{1}{m!}\left(A_1 + (-1)^m A_2\right) \tau^m| < R_{2,m} \tag{42}$$

Notice that for all $m > 1$

$$A_1 + (-1)^m A_2 = A_1 + (-1)^{m-2} A_2.$$

Dropping the remainder in equation 42 gives us

$$\langle F, U_m \rangle = \frac{1}{m!} (A_1 + (-1)^m A_2) \tau^m. \tag{43}$$

From dividing equation 43 with it self for two different indexes m and $m-2$ we get that

$$\tau = \sqrt{\frac{\langle F, U_m \rangle m (m-1)}{\langle F, U_{m-2} \rangle}}. \tag{44}$$

Notice also that we can verify the assumption of the targets symmetry around zero by checking that for all $l, m > 1$

$$\frac{\langle F, U_m \rangle m (m-1)}{\langle F, U_{m-2} \rangle} = \frac{\langle F, U_l \rangle l (l-1)}{\langle F, U_{l-2} \rangle}. \tag{45}$$

Now in general situation (non symmetric targets) we can proceed as follows: we shift our system to the time t_0 where our measurement β_i satisfy 45. The position of the shift will satisfy

$$t_0 = \frac{\tau_1 + \tau_2}{2} \tag{46}$$

and by denoting

$$\tau = \frac{\tau_1 - \tau_2}{2} \tag{47}$$

we will get a symmetric situation. Then we can find τ using 44. In order to find A_1, A_2 we will solve the first two equations for γ_1, γ_2 with A_1, A_2 as unknowns and τ substituted from 44. We will solve the system

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \gamma_0 \\ \frac{\gamma_1}{\tau} \end{pmatrix} \tag{48}$$

which is solvable since its determinant is $-2 \neq 0$.
Since τ is small as m gets larger the expression $\frac{1}{m!} (A_1 + (-1)^m A_2) \tau^m$ gets smaller, so we will get the best results from the few first equations.

For the case $m = 2, 3$ we get explicitly:

$$\tau = \sqrt{\frac{\langle F, U_2 \rangle 2}{\langle F, U_0 \rangle}}$$

as a solution for τ and 45 becomes

$$\frac{\langle F, U_2 \rangle}{\langle F, U_0 \rangle} = 3 \frac{\langle F, U_3 \rangle}{\langle F, U_1 \rangle}.$$

For numerical results concerning this section see §4.8.6.

4.8 Numerical results.

4.8.1 Introduction.

Below we present some initial numerical results in the colliding target problems. The main purpose of the numerical simulations we have tested was to check a feasibility of the suggested approach. We still do not address some important issues in the practical implementation of the CTD problem.
Each section contains a detailed explanation of the results presented in it.
At the end of each section the reader could see a table summarizing all the results together.

4.8.2 General explanation about the simulations

In each simulation we generated a function and a random noise. If the noise amplitude was as high as the amplitude of the function we call the function: *signal*. If the noise amplitude was much smaller than the amplitude of the function we call the function: *convolution mask*.
In each section we superposed the function with it self after shifting it and multiplying it with different amplitudes. To this superposition we added the noise with different amplitudes. Then in each section we calculated the shifts and the amplitude using different methods.
If the noise amplitude was 0 we used only one calculation as the result. If the noise amplitude was not 0 we calculated the shifts and the amplitudes 20 times. Each time with the same superposed functions but with different random noise. The result we took was the calculated mean and the standard deviation of the 20 simulations.
The noise was taken as a random noise on the interval $[-1, 1]$ or a normal (gaussian) noise with mean 0 and standard deviation 1.

4.8.3 Detecting the shift of one target from highly noisy measurements.

In this section we simulated a received signal reflected from only one target but with high noise level.

As a first transmitted signal we will have

$$M_1(t) = \frac{1}{0.2^4}\chi_{[\pi-0.2,\pi+0.2]}(t)\,(t-(\pi-0.2))^2\,(t-(\pi+0.2))^2 \tag{49}$$

see figure 1.

This is a 4 times differentiable bump, centered at π, with width of 0.2 to each

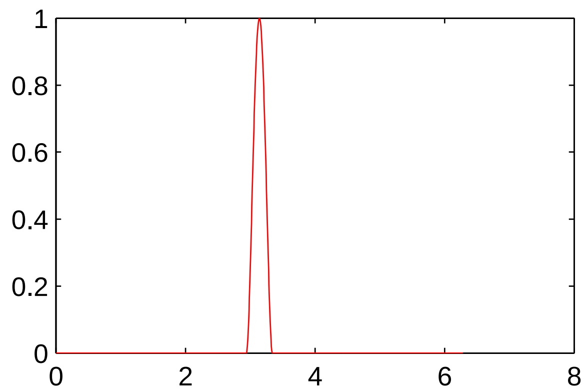

Figure 1: Transmitted bump signal without noise.

side and amplitude 1 unit. In this case the shift is not small since there is only one target. We will find the shift from the maxima of the second convolution of the received signal with the transmitted signal.

The aim in this simulation is to show that the second convolution mask decreases the noise in a satisfactory manner.

We will use a uniform noise with amplitude of 1 unit (strong as the signal it self) - see figure 2. The second convolution mask for this signal can be seen in figure 3

After averaging 20 simulations of the second convolution mask of this signal with different uniform noises and with a shift of 2 units we got that the averaged shift was 1.9831 with standard deviation of 0.0129.

Making the same calculations with normal noise the results were: averaged shift of 1.9092 with standard deviation of 0.4237.

As a second transmitted signal we will have

$$M_2(t) = \sin\left(t^2\right) \tag{50}$$

see figure 4.

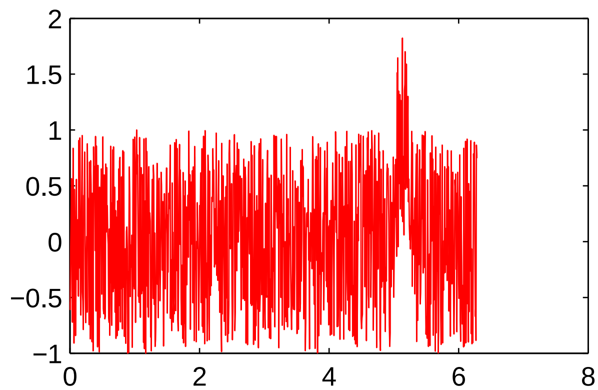

Figure 2: Transmitted bump signal shifted by 2 units with uniform noise having amplitude of 1 unit.

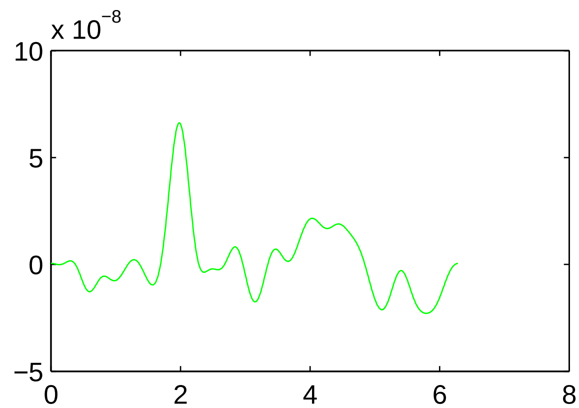

Figure 3: Second convolution mask for bump signal shifted by 2 units with uniform noise having amplitude of 1 unit .

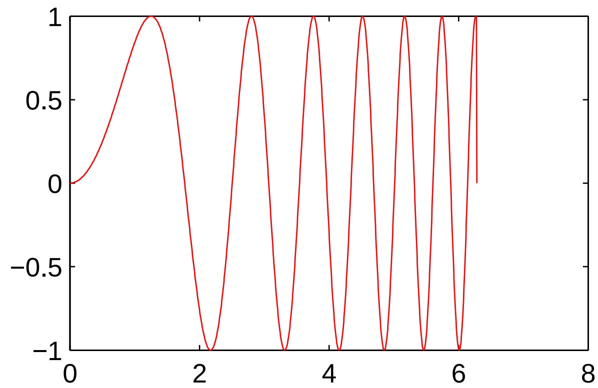

Figure 4: Transmitted $\sin(t^2)$ signal without noise.

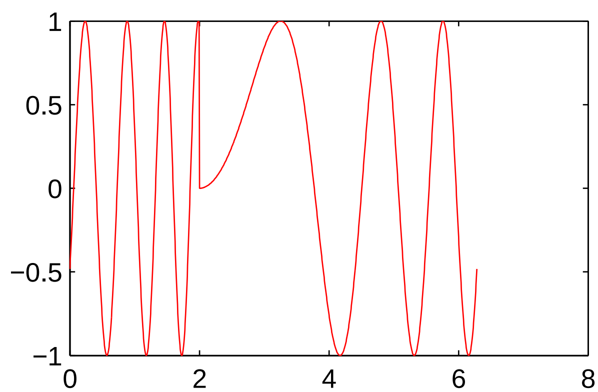

Figure 5: Transmitted $\sin(t^2)$ signal shifted by 2 units without noise - use of periodicity.

Notice that in this case the shifted signal will use 2π periodicity - see figure 5.

We will use a normal noise with amplitude of 2 units (strong as the signal it self) see figure 6.

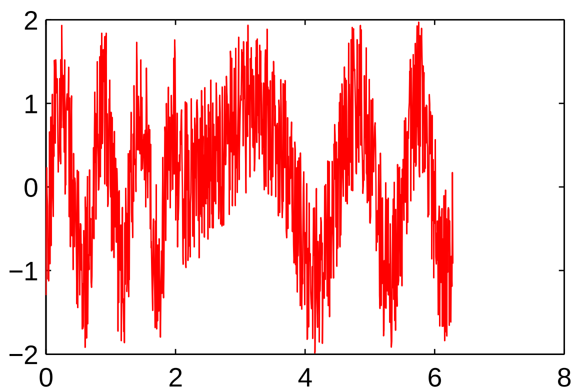

Figure 6: Transmitted $\sin(t^2)$ signal shifted by 2 units with uniform noise having amplitude of 2 units.

The second convolution mask for this signal can be seen in figure 7

After averaging 20 maximums of the second convolution mask of this signal with different uniform noises and with a shift of 2 units we got that the averaged shift was 2.0013 with standard deviation of 0.0066.

Making the same calculations with normal noise the results were: averaged shift of 2.5050 with standard deviation of 1.4615. (We do not know why in this case the calculated mean of the shift is much more far from the given shift than in other functions and noises type. This issue should be studied in a future work.)

Having these results we can see that the second convolution mask gives good results for the maxima detection. Where as can be seen from figures 2 and 6 it is more difficult to find the shift directly from them.

All the results for this section are collected in this table:

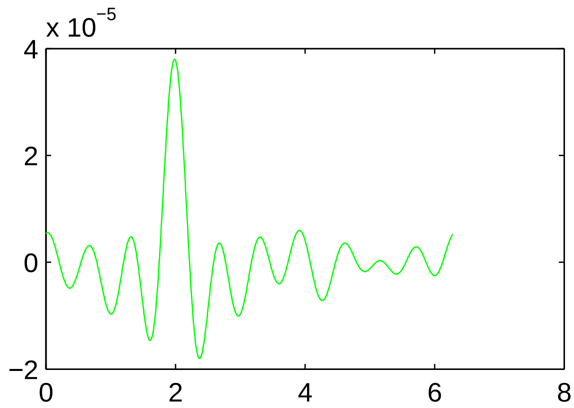

Figure 7: Second convolution mask for $\sin(t^2)$ signal shifted by 2 units with uniform noise having amplitude of 2 unit.

signal type	data type	shift
M_1	given	2
uniform noise amp=1	calculated average	1.9831
20 measurements	standard deviation	0.0129
M_1	given	2
normal noise amp=1	calculated average	1.9092
20 measurements	standard deviation	0.4237
M_2	given	2
uniform noise amp=2	calculated average	2.0013
20 measurements	standard deviation	0.0066
M_2	given	2
normal noise amp=1	calculated average	2.5050(!)
20 measurements	standard deviation	1.4625(!)

4.8.4 Solution for two targets with one amplitude.

In this section we will show some numerical results of two targets detection with the same amplitude, a small symmetric shift and noise.

Our first convolution mask was

$$G_1(t) = \left(t^2 - 1\right)^2 \chi_{[-1,1]}(t)$$

which is a symmetric 4 times differentiable bump on the interval $[-1, 1]$ with amplitude of 1 unit. See figure 8

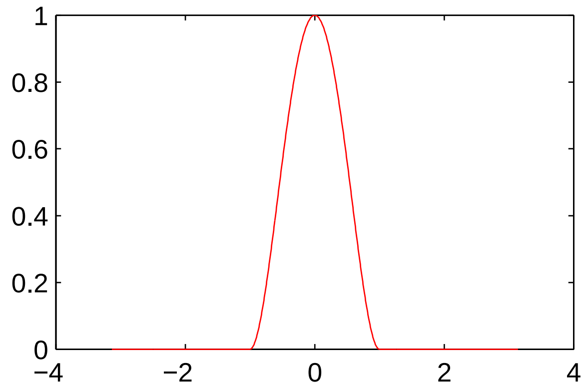

Figure 8: A bump signal without noise.

Next we Superposed two signals with the same amplitude of 0.5 unit and shifted symmetrically around zero by 0.15 unit each. See figure 9. We can also see in this figure that the superposed signal has a shape similar to the original function.

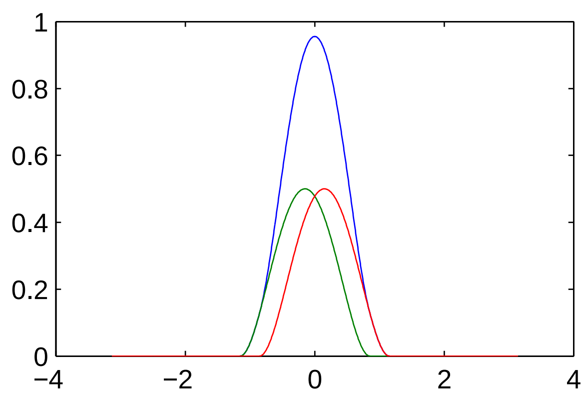

Figure 9: Two bump signals without noise shifted by 0.15 each with common amplitude 0.5 units and their superposition.

Trying to calculate the shift in this case using expressions 18 and 17 above gives us a shift of 0.138 and amplitude of 0.4977.

Using smaller shift of 0.01 gave much better results of 0.0099 shift and 0.5 amplitude.

With small shifts we could calculate the shift and the amplitude also in the presence of noise as long as the amplitude of the noise was at least one magnitude of order smaller than the shift.

Our second convolution mask was

$$G_2\left(t\right) = \sin^2\left(t^2 - 1\right)\chi_{[-1,1]}(t)$$

See figure 10.

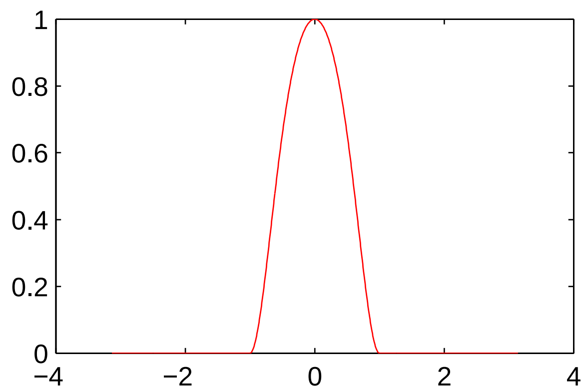

Figure 10: A $\sin^2\left(t^2 - 1\right)$ signal without noise.

Next we Superposed two signals with the same amplitude of 0.5 unit and shifted symmetrically around zero by 0.15 unit each. See figure 11.

Trying to calculate the shift in this case using expressions 18 and 17 gives us a shift of 0.1328 and amplitude of 0.4969.

Using smaller shift of 0.01 gave much better results of 0.0099 shift and 0.5 amplitude.

With small shifts we could calculate the shift and the amplitude also in the presence of noise as long as the amplitude of the noise was at least one magnitude of order smaller than the shift.

All the results for this section are collected in this table:

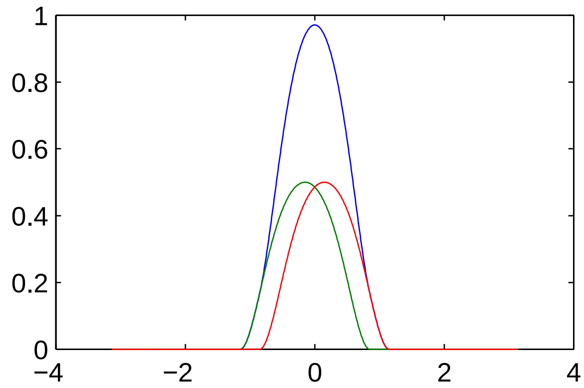

Figure 11: Two $\sin^2(t^2-1)$ signals shifted by 0.15 each, with common amplitude 0.5 units without noise and their superposition.

signal type	data type	amplitude	shift
G_1	given	0.5	0.15
	calculated	0.4977	0.1380
G_1	given	0.5	0.01
	calculated	0.5	0.0099
G_2	given	0.5	0.15
	calculated	0.4969	0.1328
G_2	given	0.5	0.01
	calculated	0.5	0.0099

4.8.5 Solution for two targets with two different amplitudes.

In this section we shall find A and τ in formula 23 using expressions 27 and 28. Our first simulation will use convolution mask of the form

$$G_3(t) = \left(t^2-1\right)^2 \chi_{[-1,1]}(t).$$

This is a four time differential bump on the interval $[-1,1]$ with amplitude of 1 units. Next we generated 23 with $A=2$ and $\tau=0.1$ without noise. Using 27 and 28 we got shift of 0.0968 units and amplitude of 1.9994 units. With presence of uniform noise of amplitude 0.1 (as strong as the signal it self) after averaging 20 results (from different uniform noises) we got averaged shift of 0.0940 units with standard deviation of 0.0221 units, and amplitude of 1.9979 units with standard deviation of 0.0059 units.

Our second simulation will use convolution mask of the form

$$G_4(t) = \sin^2\left(t^2 - 1\right)\chi_{[-1,1]}(t).$$

Next we generated 23 with $A = 2$ and $\tau = 0.1$ without noise. Using 27 and 28 we got shift of 0.0956 units and amplitude of 1.9992 units. With presence of uniform noise of amplitude 0.1 (as strong as the signal it self) after averaging 20 results (from different uniform noises) we got averaged shift of 0.0948 units with standard deviation of 0.0140 units, and amplitude of 1.9995 units with standard deviation of 0.0050 units.
All the results for this section are collected in this table:

signal type	data type	amplitude	shift
G_3	given	2	0.1
	calculated	1.9994	0.0968
G_3	given	2	0.1
Uniform noise amp=0.1	calculated average	1.9979	0.0940
20 Measurements	standard deviation	0.0059	0.0221
G_4	given	2	0.1
	calculated	1.9992	0.0956
G_4	given	2	0.1
Uniform noise amp=0.1	calculated average	1.9995	0.0948
20 Measurements	standard deviation	0.0050	0.0140

4.8.6 Solution using 44 and 48.

In this section we will solve 36 for the case $N = 2$ and under the assumption that the system is symmetric i.e $\tau_1 = -\tau_2$, using 44 and 48. We generate 36 as an outcome of a symmetric CTD problem with two targets and different amplitudes. Our first simulation will use signal of the form

$$M_3(t) = \left(t^2 - 1\right)^4 \chi_{[-1,1]}(t).$$

This is an eight time differential bump on the interval $[-1, 1]$ with amplitude of 1 units. Next we superposed two signals with amplitude 2 and 1 units and shifts of 0.1 to each side of zero without noise. Using 44 and 48 we got shift of 0.09938 units and amplitudes of 0.99674 and 2.00249 units. With presence of uniform noise of amplitude 1 (as strong as the signal it self) after averaging 20 results (from different uniform noises) we got averaged shift of 0.09502 units with standard deviation of 0.02163 units, and amplitudes of 0.94251 and 2.04236 units

with standard deviation of 0.16685 and 0.15867 units.
Our second simulation will use signal of the form

$$M_4(t) = \left(t^2 - 1\right)^4 e^{it^2} \chi_{[-1,1]}(t).$$

For this signal the calculations will lead to complex solutions. We will take the real part of the solution since it is the closest real number to the solution (in all reasonable norms).
Next we superposed two signals with amplitude 2 and 1 units and shifts of 0.1 to each side of 0 without noise. Using 44 and 48 we got shift of 0.09919 units and amplitudes of 0.99575 and 2.00322 units. With presence of uniform noise of amplitude 1 (as strong as the signal it self) after averaging 20 results (from different uniform noises) we got averaged shift of 0.09797 units with standard deviation of 0.01689 units, and amplitudes of 0.97988 and 2.01596 units with standard deviation of 0.16409 and 0.14702 units.
All the results for this section are collected in this table:

signal type	data type	amplitude1	amplitude2	shift
M_3	given	2	1	0.1
	calculated	2.00249	0.99674	0.09938
M_3	given	2	1	0.1
Uniform noise amp=1	calculated average	2.04236	0.94251	0.09502
20 Measurements	standard deviation	0.15867	0.16685	0.02163
M_4	given	2	1	0.1
	calculated	2.00322	0.99575	0.09919
M_4	given	2	1	0.1
Uniform noise amp=1	calculated average	2.01596	0.97988	0.09797
20 Measurements	standard deviation	0.14702	0.16409	0.01689

5 Conclusions.

In this section we will overview all the sections we had through this work. We shall point the most significant topics and results we had and suggest directions for future discussion.

5.1 Non Linear Fourier and Moment Reconstruction.

- In section §2 we used the notion of *semi-algebraic complexity* in order to show that for simple functions, and in particular *piecewise polynomials*, the *partial Fourier transform* is injective. We showed the same result for the *moment transform*. This injectivity is a necessary condition for reconstructing a function from its first few moments. We showed some bounds for the number of moments needed in order to get the injectivity.
- At the end of section §2 we suggested that it is possible to find generalized polynomials from an L^2 basis with a certain sign pattern. This topic should be studied in a future work. First for classic bases in L^2 as Legendre, Hermite, Laguerre, etc. Second to find general conditions on an L^2 basis to posses this property.
- In section §3 we formed a set of non linear equations. We believe that the solution of this set will be a good approximation for the parameters of the function we need to reconstruct. This claim should be studied in a future work.
- The set of equations, 36, that rose from the piecewise constants function's reconstruction is the same set that rose from the *CTD problem* we analyzed in later section. Other sets of equations that rose from general piece-wise polynomial function have similar form as the one we studied in this work. The solutions of these sets should be studied in a future work.

5.2 The Colliding Targets Detection Problem.

- In section §4 we described the problem of target detection and in particular the *colliding target detection - the CTD*.
- In §4.3 we discussed about *noise reduction* and the *convolution mask*. We used these two notions in order to find maxima of a noisy signal.
- In section §4.4 we analyzed a simplified case of CTD. We had 2 targets with the same reflection amplitude shifted symmetrically around zero. The shift and the amplitude were the parameters in this problem. We used second

degree *Taylor approximation* of the signal and some *convolution kernels* to find the parameters.

- In section §4.5 we analyzed two targets with two different amplitudes and shifts.
- In section §4.6 we analyzed the general CTD problem and generated a set of non linear equations-36, on the parameters (the unknowns amplitudes and shifts).
- In section §4.6.2 We introduced a notion of a dual basis (or kernels) for a function and its derivatives. We saw that in all reasonable cases neither the function and its derivatives nor the set of kernels are orthogonal sets. We suggested to check how close to orthogonal we can bring them. This suggestion should be studied in a future work.
- In section §4.7 we showed some suggested solutions of the set 36. Also we used 36 to solve a two colliding targets case. These solutions of 36 should be studied in a future work.
- In section §4.8 we showed some initial numerical results intended for a basic testing of our approach. All the numerical schemes and simulations should be studied in a future work, in much more details

5.3 General Remarks.

This work opens several direction for future research. Most of the numerical schemes, estimations, noise analysis and some more topics in this work should be studied in a future work.

List of Figures

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