Decoupling of Fourier Reconstruction System for Shifts of Several Signals

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Weizmann Institute of Science Department of Mathematics Rehovot, Israel.

July 5th 2013
Sampta 2013
Jacobs university - Bremen Germany

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partnering for process control

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Lecture outline.

1. Algebraic Sampling - FRI - Prony System
   - The main problem
   - FRI - The Prony system
2. Fourier decoupling - Shifts of several functions.
   - Sampling - The freedom of choice
   - The Quasi Prony system
   - A 2-dimensional example
   - A 1 dimensional simulation

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“Algebraic Sampling” or “Algebraic Signal Reconstruction”.

$$F = F_p(x),\ x \in \mathbb{R}^d$$

$p = (p_1, \ldots, p_r)$ a set of parameters.

The main problem is:

How to reconstruct in a robust and efficient way the parameters p from a set of “measurements” $m_1(F), \ldots, m_n(F)$*?*

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- Substitute F in the symbolic expression of the measurements (Fourier$\backslash$Moments integrals).
- Equate the resulting symbolic expressions to the measurements $m_n(F)$ to get a (non linear) set of equations of the parameters.
- Solve the equations to find the parameters.
- We, mainly, set and solve the ”Prony system”.

The main Problem.

We focus on *shift-generated signals*, F, of the form:

$$F(x) = \sum_{i=1}^{M} \sum_{j=0}^{L} \sum_{k=1}^{N} A_{i,j,k} f_i^{(j)}(x - x_{i,j,k}), \; x \in \mathbb{R}^d. \tag{1}$$

and measurements

$$m_n(F) = \int F(x) x^n dx. \tag{2}$$

Here $n = (n_1, \dots, n_d) \in \mathbb{N}^d$ is a multi-index and $x^n = \prod_{s=1}^{d} x_s{}^{n_s}$.

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The 1 dimensional case - FRI - The Prony system.

As a first case we shall consider $d = 1$, $M = 1, L = 0,\ f(x) = \delta(x)$

$$F(x) = \sum_{k=1}^{N} A_k \delta(x - x_k) \tag{3}$$

and

$$m_n(F) = \int F(x) x^n dx. \tag{4}$$

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Substituting F into the integral formula of $m_n(F)$ will give us the Prony system.

The Prony system.

Le Baron Gaspard Clair Franaçois Marie Riche de Prony 1795-1839

$$m_n = \sum_{k=1}^{N} A_k x_k^n, \; n \in \mathbb{N} \tag{5}$$

This system, of nonlinear equations, appears in several theoretical areas as well in different important applications.

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Solution methods:

Annihilating filters $\sim$ Padé approximation $\sim$ Prony method.

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Fourier decoupling - Shifts of several functions.

$$F(x) = \sum_{i=1}^{M} \sum_{k=1}^{N_i} A_{i,k} f_i(x - x_{i,k}),\ x,\ x_{i,k} \in \mathbb{R}^d. \tag{6}$$

The models $f_1, \ldots, f_{N_i}$ are given. (in particular, their Fourier transforms $\hat{f}_i(\omega)$ are known.)

$A_{i,k},\ x_{i,k}$ are the unknown signal parameters to be found.

Notice that here $M \geq 2$.

Decoupling strategy - Sampling - "The Freedom of choice"

- Fourier coefficients (moments) of $F \longrightarrow$ "non-uniform samples" of the Fourier (Mellin) transforms of F : $c_s(F) = \int_{\mathbb{R}^d} e^{-i<x,s>} F(x)dx$.
- A well chosen "sampling set" $Z \subset \mathbb{R}^n$ to decouple the system

$$\sum_{i=1}^{M} \sum_{k=1}^{N_j} A_{i,k} c_{s_l}(f_i) y_{i,k}^{s_l} = c_{s_l}(F), \ l = 1, \ldots, m \tag{7}$$

where $y_{i,k}^{s} = e^{-i<x_{i,k},s>}$.

- Typically, the sampling set Z will be finite.

Sampling in Fourier space.

$$Z_i = \{s : \hat{f}_i(s) = 0\}, i = 1, \ldots, m$$

$$Z = \{s_1, \ldots, s_l, \ldots\} \subset (\cap_{i' \neq i} Z_{i'}) \setminus Z_i.$$

Then system (7) takes the following, "Quasi Prony" type, form

$$\sum_{k=1}^{N_i} A_{i,k} y_{i,k}^{s_l} = C_{s_l}(F), \; l = 1, 2, \ldots \; where \; C_{s_l}(F) = \frac{c_{s_l}(F)}{c_{s_l}(f_i)}. \tag{8}$$

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- Generically: if $M \leq d + 1$ then Z_i are hypersurfaces of dimension $d - 1$ intersecting transversally and we can, at least theoretically, find enough points in Z for our decoupled Prony type equations.

The Quasi Prony system

Equation (8)

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will be of a Prony type if $s_l = r_0 + r_1 \cdot l$ other wise we call it a quasi-Prony system.

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will be of a Prony type if $s_l = r_0 + r_1 \cdot l$ other wise we call it a quasi-Prony system.

- The solvability and stability of the quasi type equation is analysed utilizing some geometric characterization , Turan Nazarov inequality, Khovanski's fewnomial's bounds and an innovative approach enabling reconstructing the parameters from finite set of samples

A 2-dimensional example:

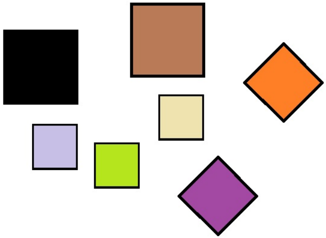
Let Q_1, Q_2 and Q_3 be the three squares

$$Q_1 = [-3,3]^2, Q_2 = [-5,5]^2,$$

$$Q_3 = \text{a rotation of the square } [-\sqrt{2},\sqrt{2}]^2 \text{ by } 45^0.$$

Let χ_i, $i = 1,2,3$ be the characteristic function of these squares:

$$\chi_i(x) = \begin{cases} 1 & x \in Q_i \\ 0 & x \notin Q_i. \end{cases}$$



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and let the signal F be

$$F(x,y) = \sum_{i=1}^{3} \sum_{k=1}^{N_i} A_{i,k} \chi_{Q_i}(x - x_{k,i}, y - y_{i,k}).$$

A 2-dimensional example:

The Fourier transforms of χ_1, χ_2 and χ_3 are

$$\hat{\chi}_1(\omega,\rho) = 4\frac{\sin 3\omega}{\omega}\cdot\frac{\sin 3\rho}{\rho} \quad \hat{\chi}_2(\omega,\rho) = 4\frac{\sin 5\omega}{\omega}\cdot\frac{\sin 5\rho}{\rho}$$

$$\hat{\chi}_3(\omega,\rho) = 8\frac{\sin\frac{\omega+\rho}{2}}{\frac{\omega+\rho}{2}}\cdot\frac{\sin\frac{\omega-\rho}{2}}{\frac{\omega-\rho}{2}}.$$

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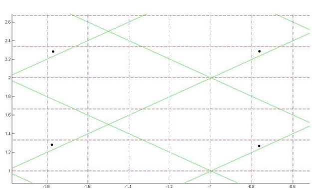
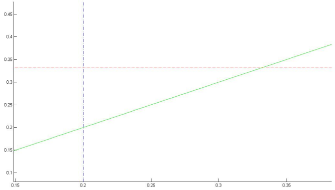


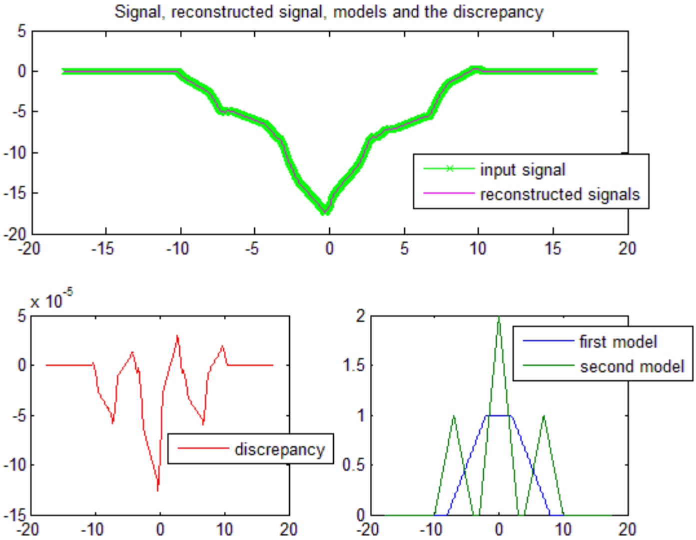
(a) Intersection of the zero sets.



(b) The repeating periodically triangle.

Figure: The Zero sets of the three signals on the (ρ,ω) Fourier space.

A 1 dimensional simulation



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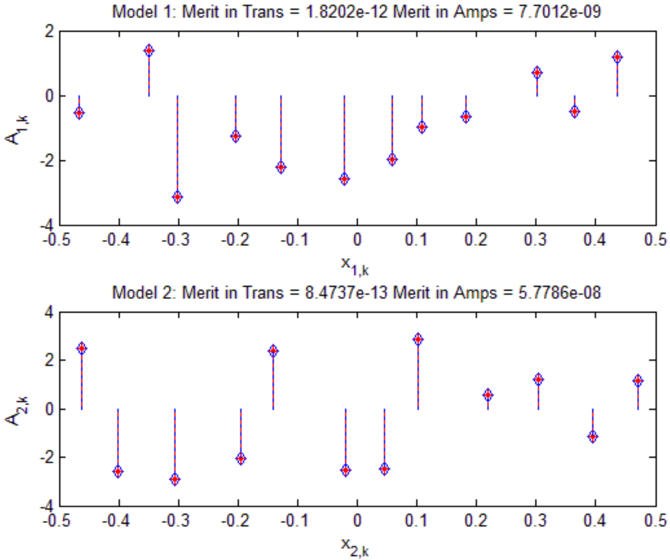

Summary

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Thank you very much for your attention.

Interpolation and Turan sets.

Definition

A set $S = \{s_1, \ldots, s_m\} \subset \mathbb{R}^n$ is called an interpolating set (IS) for exponential polynomials of degree N if any $\Phi(s) = \sum_{q=1}^{N} a_q e^{x_q s}$ with $a_q \neq 0,\ q = 1, \ldots, N$ is uniquely defined by its values on S.

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A result from the FRI/1 Dim case:

Theorem

The set of the integer points $\{0, 1, 2, \dots, 2N\}$ is an interpolating set for exponential polynomials of degree N satisfying the assumption that all the coordinates x_i are pairwise distinct.

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- similar result exists also for higher dimensions (see [S. PhD 2011]).

Bounding exp. poly.s from its values on a subset only.

Definition

A set $S \subset \mathbb{R}^n$ is called a Turan set (of degree N) if for each poly-interval $I^n \subset \mathbb{R}^n$ there is a constant $K = K_{I,S}$ such that for any exponential polynomial $\Phi(s) = \sum_{q=1}^{N} A_q e^{\lambda_q s}$ of degree N the following inequality holds:

$$\max_{I^n} |\Phi(s)| \leq K_{I,S}\, e^{\mu_n(I_n) max|Re\lambda_q|} \max_{S} |\Phi(s)|. \tag{9}$$

The minimum of the constants $K_{I,S}$ in (9) is called the Turan constant (of degree N) of the couple (I, S), and it is denoted by $TC_N(I, S)$.

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Lemma

Let $S \subset \mathbb{R}^n$ be a Turan set of degree $2N$. Then S is interpolating for exponential polynomials of degree N.

Theorem (Turan-Nazarov)

Let $p(t) = \sum_{k=0}^{m} c_k e^{\lambda_k t}$ *with* $c_k, \lambda_k \in \mathbb{C}$ *be an exponential polynomial of order* $m+1$*. Let* $I \subset \mathbb{R}$ *an interval, and* E *be a measurable subset of* I *of positive measure.*

$$\sup_{t \in I} |p(t)| \leq e^{\mu(I) \cdot \max |Re \lambda_k|} \cdot \left(\frac{C\mu(I)}{\mu(E)} \right)^m \cdot \sup_{t \in E} |p(t)| \,, \qquad (10)$$

where $C > 0$ *is an absolute constant.*

See [Naz2000].

Changing the positive measure requirement with a Geometrical characterization.

Definition (Covering number)

$M(\varepsilon, S)$ is the minimal number of closed ε-intervals covering S .

Definition

Let $P_{n,N}(\varepsilon)$ be a polynomial $P_{n,N}(\varepsilon) = C_0 + C_1\frac{1}{\varepsilon} + \cdots + C_{n-1}(\frac{1}{\varepsilon})^{n-1}$ defined in [Fri.Yom].

Definition (The Metric Span)

For $S \subset \mathbb{R}^n$ $\omega_{n,N}(S) = \sup_\varepsilon \varepsilon^n[M(\varepsilon, S) - P_{n,N}(\varepsilon)]$.

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For $S \subset \mathbb{R}^n$ $\omega_{n,N}(S) = \sup_\varepsilon \varepsilon^n [M(\varepsilon, S) - P_{n,N}(\varepsilon)]$.

Actually, the metric span $\omega \leq \mu$ hence:

Theorem ([Fri.Yom])

The Lebesgue measure $\mu(S)$ in the Turan-Nazarov inequality can be replaced with $\omega = \omega_{n,N}(S)$. More specifically, for each S we have the following: Let $p(t) = \sum_{k=0}^{m} c_k e^{\lambda_k t}$ with $c_k, \lambda_k \in \mathbb{C}$ be an exponential polynomial of order N. Let $I \subset \mathbb{R}$ be an interval, and E be a measurable subset of I. Then

$$\sup_{t\in I} |p(t)| \le e^{\mu(I)\cdot \max|Re\lambda_k|} \cdot \left(\frac{C\mu(I)}{\omega(E)} \right)^m \cdot \sup_{t\in E} |p(t)| \ , \qquad (11)$$

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where $C > 0$ is an absolute constant.

Finiteness of interpolation sets.

Notice that E can be now of a zero measure or even a finite set with a non zero $\omega(E)$ hence we may have finite interpolation sets.

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Different characterization.

In the published paper we present another robust characterization for E.

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