An "algebraic" reconstruction of piecewise-smooth functions from integral measurements

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The 11th Israeli Mini-Workshop in Applied and Computational Mathematics
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Classical results and motivation.

- A C^k function can be approximated from its Fourier partial sum of length N with the error of order $\frac{C}{N^K}$.
- For functions with singularities (e.g discontinuous functions) the Fourier approximation method suffers from slow convergence and oscillations near the discontinuity point (Gibbs effect, over shoot etc).
- Using the notion of Kolmogorov's $n^{\underline{th}}$ width it can be shown that any approximation method with linear scheme can not be better than the Fourier partial sum approximation.
- To reconstruct non continuous signals from their Fourier data we must consider a non linear approximation scheme.

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Algebraic sampling

Algebraic sampling:

- A-priori assumptions: ”Simple” signals - Signals with a known structure and few degrees of freedom ($\leq n$).
- The Linear measurements should be of a given, known analytic type (e.g: Fourier coefficients, moments, samples of the signal - convolution against a known kernel, some other integral form, etc)
- Using the a - priori knowledge and the given measurements we shall reconstruct the signal with $\approx n$ measurements (n is assumed to be small).

A simple example

- The signal

$$F(x)=\sum_{j=1}^{r}A_j\delta(x-x_j),\ n=2r$$

- The measurements (moments)

$$\mu_k(F)=\int x^k F(x)dx$$

- The Prony system (the unknowns are the A_j's and the x_j's)

$$\begin{aligned} A_1+\ldots+A_r &= \mu_0(F)\\ A_1x_1+\ldots+A_rx_r &= \mu_1(F)\\ &\vdots\\ A_1x_1^{2r}+\ldots+A_rx_r^{2r} &= \mu_{2r}(F)\end{aligned}$$

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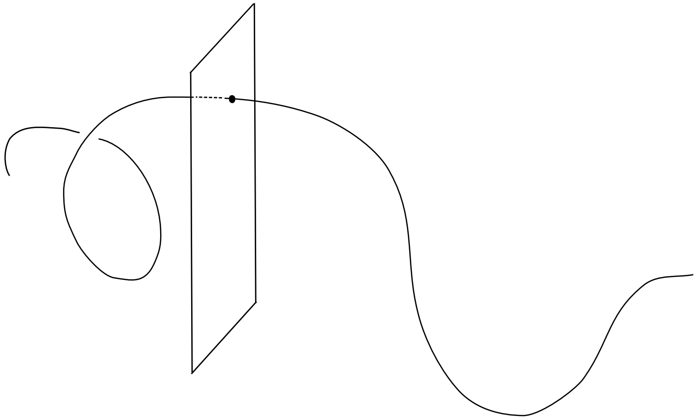
Nonlinear reconstruction through linear measurements



Dimension of the signal (n) = number of measurements

Questions

1. Can the assumption that the signal has a known "geometric" structure be justified in applications?

2. Can the arising systems be solved in a robust and efficient way?

(One of) The ultimate test(s) - General images

Question 1

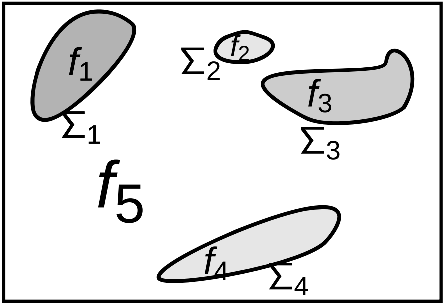
Can the assumption that the signal has a known “geometric” structure be justified in applications?

Answer

Image representation and compression via geometric models.
A very difficult problem in image precessing.

General images

For this lecture, images = Piecewise smooth functions on $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$.



Question 2

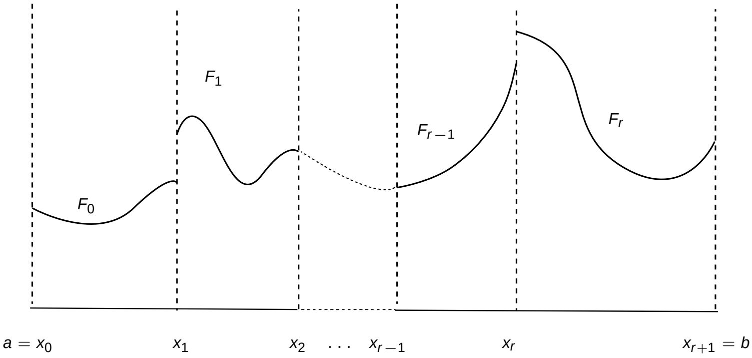
Can the arising systems be solved in a robust and efficient way?

The rest of the talk will give a partial answer to this question.

Piecewise-smooth reconstruction: related work

- [Gustafsson et al.(2000)Gustafsson, He, Milanfar, and Putinar]
- [Vetterli et al.(2002)Vetterli, Marziliano, and Blu]
- [Elad et al.(2004)Elad, Milanfar, and Golub]
- [Maravic and Vetterli(2004)]
- [Kvernadze(2004)]
- [Dragotti et al.(2007)Dragotti, Vetterli, and Blu]
- [Kisunko(2008)]
- [Yomdin and Sarig(2008)]
- [Yomdin et al.(2009)Yomdin, Batenkov, and Sarig]
- [Batenkov(2009)]

Piecewise-smooth model: a-priori information



$$\|F_j\|_{C^k} \le M < \infty, \qquad |x_{j+1} - x_j| \ge \delta > 0$$

$$\max_{l=0,\dots,k} |F_j^{(l)}(x_j) - F_{j-1}^{(l)}(x_j)| \ge \gamma > 0$$

Reconstruction accuracy

theorem

F can be reconstructed from $N + n(k, r)$ Fourier coefficients with the maximal error $\leq \frac{C}{N^k}$ both in the positions of the discontinuities and in the pointwise values of the smooth part, where $C = C(r, M, \delta, \gamma)$.

Proof:

- $$F = \overbrace{\underbrace{\tilde{F}}_{\in C^k}}^{\text{smooth part}} + \overbrace{\underbrace{\Psi}_{\text{has } n(k,r) \text{ degrees of freedom}}}^{\text{“simple” part}}$$
  $$\underbrace{\widehat{f}_N(F)}_{\text{measurement}} = \underbrace{\widehat{f}_N(\tilde{F})}_{\leq \frac{C}{N^k}} + \widehat{f}_N(\Psi)$$
- For $i \geq N$ solve the system
  $$\widehat{f}_i(\Psi) = \widehat{f}_i(F)$$
  with respect to Ψ, the “simple” part of F, with error $\leq \frac{C}{N^k}$.
- Use $\hat{f}_i(F), i \leq N$ to find $\tilde{F}$, the smooth part of F.
- Estimate the robustness.

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Reconstruction accuracy: related work

- [Eckhoff(1993)]
- [Gottlieb and Shu(1996)]
- [Gelb and Tadmor(2001)]
- [Kvernadze(2004)]

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The simple example from the beginning of the talk

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Solution of Prony system

- Setting the moments generating function:

$$I(z) = \sum_{k=0}^{\infty} \mu_k(F) z^k.$$

- $I(z) = \sum_{j=1}^{r} \frac{A_j}{1-x_j z}$ is a rational function. Its poles and the residues depend algebraically on the x_j's and the A_j's.
- Using Padé approximation method we find the poles and the residues of $I(z)$ from $2r$ of its Taylor coefficients (Our measurements).

General measurements for model reconstruction

The Model:

$$F(x) = \sum_{j=1}^{r} A_j f(x - x_j) \qquad \mu_k(F) = \int F(t)\varphi_k(t)dt$$

Given: f, $\varphi = \{\varphi_k(t)\}_{k=0}^{\infty}$, look for $\psi = \{\psi_k\}_{k=0}^{\infty}$ s.t

$$\psi_k(t) = \sum_{0\leq i\leq k} C_{i,k}\varphi_i(t) \quad and \quad \int f(t+x)\psi_k(t) = \varphi_k(x)$$

The ψ is called an "f-convolution dual" sequence of functions (similar to a bi-orthogonal set of function) with respect to the system φ.

Theorem

Let a sequence $\psi = \{\psi_k(t)\}_{k=0}^{\infty}$ be an f-convolution dual to φ. Define the generalized moments by $M_k = \sum_{0 \le i \le k} C_{i,k}\mu_i$. Then the parameters A_j and x_j in the model satisfy the following system of equations ("generalized Prony system"):

$$\sum_{j=1}^{r} A_j \varphi_k(x_j) = M_k, \quad k = 0, 1, \ldots .$$

The generating function for several cases

Multi dimensional signal reconstruction.

The model:

$$F(x) = \sum_{j=1}^{r} A_j f(x - x_j).$$

The convolution dual to the monomials $\varphi_k(x) = x^k$ are some specific polynomials $\psi_k(x)$.
The generalized moments generating function:

$$I(z) = \sum_{j=1}^{r} A_j \prod_{l=1}^{d} \frac{1}{1 - (x_j)_l z_l}, x_j \in \mathbb{R}^d$$

The generating function for several cases

Adding the derivatives of the given function f

The model:

$$F(x) = \sum_{j=1}^{r} \sum_{l=0}^{s} A_{j,l} f^{(l)}(x + x_j).$$

We use the same polynomials $\psi_k(x)$ as in the previous case.
The generalized moments generating function :

$$I(z) = \sum_{j=1}^{r} \sum_{l=0}^{s} \sum_{q=0}^{l} \binom{l}{q} \frac{(-1)^{q+l} A_{j,l}/(x_j)^l}{(1 - x_j z)^{q+1}}, \; x_j \in \mathbb{R}$$

Reconstruction of a signal built from two functions

The modified model:

$$F(x) = \sum_{j=1}^{r} a_j f(x + x_j) + \sum_{j=1}^{r} b_j g(x + y_j)$$

where f and g are two different given functions and the a_j's, b_j's, x_j's and the y_j's are the unknown parameters to be reconstructed from finitely many linear measurements.

Assumption on f and g :

There are infinitely many zeros of the Fourier transform $\hat{f}$ which are not zeros of the Fourier transform $\hat{g}$ and vice versa.

This gives us:

Generalized Prony system

$$\gamma_k = \sum_{j=1}^{r} a_j (x_j)^{\omega_k}$$

where ω_k are some of the zeros of the Fourier transforms of f or g. They are not necessarily integers (as in the usual Prony system).

Solvability of this system depends on the geometry of the points ω_k.

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Example in dimension 1

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$$g(x) = \delta(x+1) + \delta(x-1)$$

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- $f(x) = H_{[-1,1]}(x),\ g(x) = \delta(x+1) + \delta(x-1).$
- $\hat{f}(\omega) = \sqrt{\frac{2}{\pi}}(\sin\omega)/\omega,\ \hat{g}(\omega) = \sqrt{\frac{2}{\pi}}\cos\omega.$
- $Z(\hat{f}) = \pi\mathbb{Z}/\{0\}, Z(\hat{g}) = \pi(\mathbb{Z}+\frac{1}{2}),$
  $|Z(\hat{f})| = |Z(\hat{g})| = \infty, Z(\hat{f}) \cap Z(\hat{g}) = \emptyset.$
- The generalized Prony is actually a Prony system since the geometry of the zeros is the geometry of $\mathbb{Z}$.

$$\frac{\hat{F}(\pi n)}{\sqrt{\frac{2}{\pi}}(-1)^n} = \sum_{j=1}^{r} a_j((x_j)^{\pi})^n, n \in \mathbb{Z}/\{0\}$$

$$\frac{\hat{F}((\frac{1}{2}+n)\pi)}{\sqrt{\frac{2}{\pi}\frac{(-1)^{n+1}}{(\frac{1}{2}+n)\pi}}} = \sum_{l=1}^{r}(b_l(y_l)^{\frac{\pi}{2}})((y_l)^{\pi})^n \ , n \in (\mathbb{Z}+\frac{1}{2})$$

This example can be generalized to multidim. signals.

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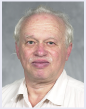
This is a joint work with


Dima Batenkov


Yosef Yomdin

At the Weizmann institute of Science.

Thank you for listening.

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